Hausdorff measure on o-minimal structures

Version 3.6

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Abstract

We introduce the Hausdorff measure for definable sets in an o-minimal structure, and prove the Cauchy-Crofton and co-area formulae for the o-minimal Hausdorff measure. We also prove that every definable set can be partitioned into "basic rectifiable sets", and that the Whitney arc property holds for basic rectifiable sets.

Keywords: O-minimality, Hausdorff measure, Whitney arc property, Cauchy-Crofton, coarea.

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1 Introduction

Let K be an o-minimal structure expanding a field. We introduce, for every $e \in \mathbb{N}$, the e-dimensional Hausdorff measure for definable sets, which is the generalization of the usual Hausdorff measure for real sets [Morgan88]. We also show that every definable set can be partitioned into "basic e-rectifiable sets" (§3). Moreover, we generalize some well known result from geometric measure theory, such as the Cauchy-Crofton formula (which computes the Hausdorff measure of a set as the average number of points of intersection with hyperplanes of complementary dimension) and the co-area formula (a generalization of Fubini's theorem), to the o-minimal context.

The measure defined in [BO04] is the starting point for our construction of the Hausdorff measure. A theorem of [BP98] allows us to prove that integration using the Berarducci-Otero measure satisfies properties analogous to the ones for integration over the reals (for example, the change of variable formula). If K is sufficiently saturated, the Berarducci-Otero measure of a bounded definable set X is $\mathcal{L}_{\mathbb{R}}(\mathrm{st}(X))$, where $\mathcal{L}_{\mathbb{R}}$ is the Lebesgue measure and st is the standard-part map. However, the naive definition of Hausdorff measure given by

$$\mathcal{H}^e(X) := \mathcal{H}^e_{\mathbb{R}}(\mathrm{st}(X)) \tag{1}$$

does not work (because the resulting “measure” is not additive: see Example 5.8). The correct definition for the e-dimensional Hausdorff measure is defining it first for basic e-rectifiable sets via (1), and then extending it to definable sets by using a partition into basic e-rectifiable pieces. Such a partition is obtained by using partitions into M_n-cells ([K92], [P08], [VR06]), a consequence of which is the Whitney arc property for basic e-rectifiable sets (§4).

2 Lebesgue measure on o-minimal structures

The definitions of measure theory are taken from [Halmos50].

Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be the extended real line. Let K be a $\aleph_1$-saturated o-minimal structure, expanding a field. Let $\mathring{K}$ be the set of finite elements of K. Let $\mathrm{st} : K^n \to \bar{\mathbb{R}}^n$ be the function mapping $\bar{x}$ to the n-tuple of standard parts of the components of $\bar{x}$.

For every $n \in \mathbb{N}$, let $\mathcal{L}^n_{\mathbb{R}}$ be the n-dimensional Lebesgue measure (on $\mathbb{R}^n$). If n is clear from context we drop the superscript. Let $\mathcal{L}^n_1$ be the o-minimal measure on $\mathring{K}^n$ defined in [BO04]. More precisely, $\mathcal{L}^n_1$ is a measure on the σ-ring R_n generated by the definable subsets of $\mathring{K}^n$; thus, $(\mathring{K}^n, R_n, \mathcal{L}^n_1)$ is a measure space. Moreover, since $\mathring{K}^n \in R_n$, R_n is actually a σ-algebra.

Notice that $\mathcal{L}^n_1$ can be extended in a natural way to a measure $\mathcal{L}^n_2$ on the σ-ring $\mathcal{B}_n$ generated by the definable subsets of K^n of finite diameter. Finally, we denote by $\mathcal{L}^n$ the completion of $\mathcal{L}^n_2$, and if n is clear from context we drop the superscript. Notice that the σ-ring $\mathcal{B}_n$ is not a σ-algebra.

Remark 2.1 ([BO04, Thm. 4.3])**.** If $C \subset \mathring{K}^n$ is definable, then $\mathcal{L}^n(C)$ is the Lebesgue measure of $\mathrm{st}(C)$.

Definition 2.2. For $A \subseteq K^n$ and $f : K^n \to K^m$ we define $\mathrm{st}(f) : A \to \bar{\mathbb{R}}^m$ by $\mathrm{st}(f)(x) = \mathrm{st}(f(x))$.

Remark 2.3. If $A \subseteq \mathring{K}^n$ and $f : A \to K$ are definable, then $\mathrm{st}(f)$ is an $\mathcal{L}^n$-measurable function.

Definition 2.4. Let $A \subseteq \mathring{K}^n$ and $f : A \to K$ be definable. If $\mathrm{st}(f)$ is $\mathcal{L}^n$-integrable we will denote its integral by

$$\int_A f \,\mathrm{d}\mathcal{L}^n; \quad \int_A f(x)\,\mathrm{d}x; \quad \int_A f(x)\,\mathrm{d}\mathcal{L}^n(x) \quad \text{or} \quad \int_A f.$$

Remark 2.5. If $A \subseteq \mathring{K}^n$ and $f : A \to \mathring{K}$ are definable, then $\mathrm{st}(f)$ is $\mathcal{L}$-integrable.

Let $\mathbb{R}_K$ be the structure on $\mathbb{R}$ generated by the sets of the form $\mathrm{st}(U)$, where U varies among the definable subsets of K^n. By [BP98], $\mathbb{R}_K$ is o-minimal.

Remark 2.6. Let $U \subseteq \mathring{K}^n$ be definable. Then, $\dim(\mathrm{st}(U)) \leq \dim(U)$.

Proof. Let $\dim(U) = d$. After a cell decomposition, we can assume that U is the graph of a definable continuous function $f : V \to \mathring{K}^{n-d}$, with $V \subset \mathring{K}^d$ open cell. We can then conclude by applying the method in [HPP08, Lemma 10.3]. □

Definition 2.7. A function f is Lipschitz if there is $C \in \mathring{K}$ such that, for all $x, y \in \mathrm{dom}(f)$, we have $|f(x) - f(y)| < C|x - y|$ (notice the condition on C being finite). An invertible function f is bi-Lipschitz if both f and f^{-1} are Lipschitz.

Remark 2.8. Let $U \subset \mathring{K}^n$ and $f : U \to \mathring{K}$ be definable, with $f \geq 0$. Then,

$$\int_U f \, \mathrm{d}\mathcal{L}^n = \mathcal{L}^{n+1}\big(\{\langle \bar{x}, y\rangle \in U \times K : 0 \leq y \leq f(\bar{x})\}\big).$$

Lemma 2.9 (Change of variables)**.** *Let $U, V \subseteq \mathring{K}^n$ be open and definable, and let $A \subseteq U$ be definable. Let $f : U \to V$ be definable and bi-Lipschitz and $g : V \to \mathring{K}$ be definable, then*

$$\int_{f(A)} g = \int_A |\det \mathrm{D}f| \; g \circ f.$$

Before proving the above lemma, we need some preliminary definitions and results.

Lemma 2.10. *Let $U \subset \mathring{K}^n$ be open and let $f : U \to \mathring{K}$ be definable. Then there is a $\mathbb{R}_K$-definable function $\overline{f} : C \to \mathbb{R}$, where $C \subset \mathrm{st}(U)$ is an open set, such that*

i) $E := \big(\mathrm{st}(U) \setminus C\big) \cup \big(C \cap \mathrm{st}(K^n \setminus U)\big)$ *is $\mathcal{L}^n_{\mathbb{R}}$-negligible (and, therefore, $\mathrm{st}^{-1}(E)$ is $\mathcal{L}^n$-negligible).*

ii) *f and $\overline{f}$ are $\mathcal{C}^1$ on $U \setminus \mathrm{st}^{-1}(E)$ and C, respectively.*

iii) *For every $x \in U$ with $\mathrm{st}(x) \in C$ we have $\mathrm{st}(f(x)) = \overline{f}(\mathrm{st}(x))$. Moreover, $\mathrm{D}f$ is finite and $\mathrm{D}(\overline{f})(\mathrm{st}\, x) = \mathrm{st}(\mathrm{D}f(x))$.*

iv)

$$\int_U f = \int_C \overline{f}.$$

Proof. By cell decomposition, we may assume that f is a function of class $\mathcal{C}^1$, and that U is an open cell. Since $\dim(\Gamma(f)) = n$, we have, by Remark 2.6, $\dim(\mathrm{st}(\Gamma(f)) \leq n$. By cell decomposition, there is an $\mathbb{R}_K$-definable, closed, negligible set $E \subset \mathrm{st}(U)$, and definable functions $g_k : \mathrm{st}(U) \setminus E \to \mathbb{R}$ of class $\mathcal{C}^1$ for $k = 1, \ldots, r$ such that $\mathrm{st}(\Gamma(f)) \cap ((\mathrm{st}(U) \setminus E) \times \mathbb{R})$ is the union of the graphs of the functions g_i. We claim that $r = 1$:
In fact, if g_1, g_2 are two different such functions, and say $g_1 < g_2$, then for some $x \in \mathrm{st}(U)$ we have $\langle x, g_1(x)\rangle$, $\langle x, g_2(x)\rangle \in \mathrm{st}(\Gamma(f))$. Since f is continuous,$\{\langle x, y\rangle : y \in (g_1(x), g_2(x))\} \subset \mathrm{st}(\Gamma(f))$. On the other hand, $\{\langle x, y\rangle : \langle x, y\rangle \in \mathrm{st}(\Gamma(f))\}$ is the finite set $\{\langle x, g_1(x)\rangle, \ldots, \langle x, g_r(x)\rangle\}$, absurd.

By [HPP08, Theorem 10.4], after enlarging E by a negligible set, we obtain i).

Let $\overline{f} := g_1$. ii) holds, and for every $x \in U$ with $\mathrm{st}(x) \in C$ we have $\mathrm{st}(f(x)) = \overline{f}(\mathrm{st}(x))$. The equality of the integrals in iv) follows from Remark 2.8. To obtain the second part of iii) we will enlarge E by a negligible set. For $i = 1, \ldots, n$ let

$$E_i := \mathrm{st}\big(\{x \in U : \frac{\partial f}{\partial x_i}(x) \notin \mathring{K}\}\big).$$

By [BP98], E_i is $\mathbb{R}_K$-definable. If $\dim(E_i) = n$, then E_i contains an open ball. This contradicts Lemma 2.5 of [BO04] by which every definable, one variable function into $\mathring{K}$ has finite derivative except on $\mathrm{st}^{-1}(A)$, for a finite set A. It follows that each set E_i is negligible and therefore, after enlarging E, we may assume that $\mathrm{D}(f)$ is finite on $U \setminus \mathrm{st}^{-1}(E)$.

It remains to prove $\mathrm{D}(\overline{f})(\mathrm{st}\, x) = \mathrm{st}(\mathrm{D}f(x))$. As before, we will enlarge E by a negligible set. Let $V := \{x \in \mathbb{R}^n : \mathrm{D}(\overline{f})(x) \neq \overline{\mathrm{D}f}(x)\}$. The set V is $\mathbb{R}_K$-definable. If V is non-negligible, then it contains an open ball and therefore w.l.o.g. we may assume that V is an open ball centered at 0. We may also assume $f(0) = 0$. After substracting from f a linear function, we can assume that $\frac{\partial f}{\partial x_i}(0) = 0$ and $\frac{\partial \overline{f}}{\partial x_i}(0) = 3\epsilon > 0$ for some index $i = 1, \ldots, n$. Therefore, on a smaller neighborhood of 0, we have $\frac{\partial f}{\partial x_i} < \epsilon$ and $\frac{\partial \overline{f}}{\partial x_i} > 2\epsilon$. Thus, for x along the x_i axis, $|f(x)| < |x|\epsilon$ and $\overline{f}(x) \geq 2|x|\epsilon$ contradicting the first part of iii), namely, $\mathrm{st}(f(x)) = \overline{f}(x)$. We conclude that V is negligible. Let E' be a negligible set such that away from $\mathrm{st}^{-1}(E')$ the equality $\mathrm{st}(\mathrm{D}f(x)) = \overline{\mathrm{D}f}(\mathrm{st}\, x)$ holds. Then away from $\mathrm{st}^{-1}(V \cup E')$ we have $\mathrm{st}(\mathrm{D}f(x)) = \overline{\mathrm{D}f}(\mathrm{st}(x)) = \mathrm{D}\overline{f}(\mathrm{st}(x))$ as wanted. By cell decomposition, E can be further enlarged so that C is open. ∎

Remark 2.11. If $f^{-1}(A)$ is negligible whenever A is, then, outside a negligible closed set, $\overline{(f \circ g)} = \overline{f} \circ \overline{g}$.

Proof of Lemma 2.9. The fact that f is bi-Lipschitz implies that $\overline{f}$ is injective (since it is also bi-Lipschitz).

Claim 1. Let $C \subset \mathrm{st}(V)$ be Lebesgue measurable. Then,

$$\mathcal{L}^n(C) = \int\limits_{(\mathrm{st}\, f)^{-1}(C)} \mathrm{st}(|\det \mathrm{D}f|).$$

In fact, by the change of variables formula (on the reals!) and Lemma 2.10,

$$\mathcal{L}^n(C) = \int\limits_{\overline{f}^{-1}(C)} |\det \mathrm{D}\overline{f}|) = \int\limits_{(\mathrm{st}\, f)^{-1}(C)} \mathrm{st}(|\det \mathrm{D}f|).$$

Claim 2. Let $h : V \to \bar{\mathbb{R}}$ be an integrable function. Then,

$$\int\limits_V h = \int\limits_U \mathrm{st}(|\det \mathrm{D}f|)\; h \circ f.$$

Claim 1 implies that the statement is true if h is a simple function. By continuity, the statement is true for any integrable function h.

In particular, we can apply Claim 2 to the function

$$h : x \mapsto \begin{cases} \mathrm{st}(g(x)) & \text{if } x \in f(A), \\ 0 & \text{otherwise,} \end{cases}$$

and obtain the conclusion. ∎

Lemma 2.12 (Fubini's theorem)**.** *$\mathcal{L}^{n+m}$ is the completion of the product measure $\mathcal{L}^n \times \mathcal{L}^m$. Therefore, if D is the interval $[0,1] \subset K$ and given $f : D^{n+m} \to D$ definable,*

$$\int\limits_{D^{n+m}} f(x,y)\, \mathrm{d}\mathcal{L}^{n+m}(x,y) = \int\limits_{D^m}\int\limits_{D^n} f(x,y)\, \mathrm{d}\mathcal{L}^m(x)\, \mathrm{d}\mathcal{L}^n(y).$$

Proof. Follows from the definition of $\mathcal{L}^n$ in [BO04]. ∎

2.1 Measure on semialgebraic sets

Definition 2.13. We say that $E \subseteq K^n$ is $\emptyset$-semialgebraic if E is definable without parameters in the language of pure fields. If $E \subseteq K^n$ is $\emptyset$-semialgebraic we denote the subset of $\mathbb{R}^n$ defined by the same formula that defines E by $E_{\mathbb{R}}$.

Remark 2.14. Let $E \subseteq \mathring{K}^n$ be $\emptyset$-semialgebraic. Then, $\mathrm{st}(E) = \overline{E_{\mathbb{R}}}$.

Let $E \subseteq K^n$ be closed and $\emptyset$-semialgebraic submanifold. Working in local charts, from [BO04] one can easily define a measure $\mathcal{L}^E$ on the σ-ring generated by the definable subsets of E of bounded diameter. We will denote in the same way the completion of $\mathcal{L}^E$. Notice that $\mathcal{L}^{K^n} = \mathcal{L}^n$.

Remark 2.15. Let E be a closed, $\emptyset$-semialgebraic submanifold of K^n of dimension e, $F := \mathrm{st}(E)$, and $C \subseteq E$ be definable and bounded. Then, $\mathcal{L}^E(C) = \mathcal{L}^F_{\mathbb{R}}(\mathrm{st}(C))$, where $\mathcal{L}^F_{\mathbb{R}}$ is the e-dimensional Hausdorff measure on F.

One could also take the above remark as the definition of $\mathcal{L}^E$ on $E \cap \mathring{K}^n$.

3 Rectifiable partitions

Theorem 3.8 shows that every definable set $A \subset \mathring{K}^n$ has a partition into definable sets which are M_n-cells after an orthonormal change of coordinates (where $M_n \in \mathbb{Q}$ depends only on n). In [P08], the author shows that a permutation of the coordinates suffices. The proof of 3.8 follows closely that of [K92]. The partition in 3.8 is then used in Corollary 3.11 to show that definable sets have a rectifiable partition.

Definition 3.1. Let $L : V \to W$ be a linear map between normed K-vector spaces. The norm of L is given by

$$\|L\| := \sup_{|v|=1} |L(v)|.$$

For V, W in the Grassmannian of e-dimensional linear subspaces of K^n, namely $\mathcal{G}_e(K^n)$, let π_V and $\pi_W \in \mathrm{End}_K(K^n)$ be the orthogonal projections onto V and W respectively. In this way we have a canonical embedding $\mathcal{G}_e(K^n) \subset \mathrm{End}_K(K^n)$. The **distance function** on the Grassmannian is given by the inclusion above:

$$\delta(V, W) := \|\pi_V - \pi_W\|.$$

For P in $\mathcal{G}_1(K^n)$ and $X \in \mathcal{G}_k(K^n)$, define

$$\delta(P, X) := |v - \pi_X(v)|,$$

where π_X is the orthogonal projection onto X, and v is a generator of P of norm 1. Note that $\delta(P, X) = 0$ if and only if $P \subset X$, $0 \leq \delta(P, X) \leq 1$ and $\delta(P, X) = 1$ if and only if $P \perp X$. Note also that $\delta(P, X)$ is the definable analogous of the sine of the angle between P and X.

Lemma 3.2. *Let $n \in \mathbb{N}_{>0}$. Then there exists an $\epsilon_n \in \mathbb{Q}_{>0}$, $\epsilon_n < 1$, such that for any $X_1, \ldots, X_{2n} \in \mathcal{G}_{n-1}(K^n)$, there is a line $P \in \mathcal{G}_1(K^n)$ such that whenever $Y_1, \ldots, Y_{2n} \in \mathcal{G}_{n-1}(K^n)$ and*

$$\begin{aligned} \delta(X_i, Y_i) < \epsilon_n, &\quad i = 1, \ldots, 2n, \quad \textit{then} \\ \delta(P, Y_i) > \epsilon_n, &\quad i = 1, \ldots, 2n. \end{aligned}$$

Proof. For $\epsilon > 0$ define $S_i(\epsilon) = \{v \in S^{n-1} : |v - \pi_{X_i}(v)| \leq 2\epsilon\}$. If $K = \mathbb{R}$, let $\epsilon_n \in \mathbb{Q}_{>0}$ be small enough so that $2n \operatorname{Vol}(S_1(\epsilon_n)) < \operatorname{Vol}(S^{n-1})$, where Vol is the measure $\mathcal{L}^{S^{n-1}}$ defined in §2.1. Then

$$\operatorname{Vol}(\bigcup_{i=1}^{2n} S_i(\epsilon_n)) \leq 2n \operatorname{Vol}(S_1(\epsilon_n)) < \operatorname{Vol}(S^{n-1})$$

and therefore

$$\bigcup_{i=1}^{2n} S_i(\epsilon_n) \neq S^{n-1}. \tag{2}$$

The same ϵ_n will necessarily satisfy (2) for any field K containing $\mathbb{R}$.

Now, we choose

$$v \in S^{n-1} - \bigcup_{i=1}^{2n} S_i(\epsilon_n)$$

and let $P := \langle v \rangle$. Then

$$\delta(P, Y_i) = |v - \pi_{Y_i} v| \geq |v - \pi_{X_i} v| - |\pi_{X_i} v - \pi_{Y_i} v| > \epsilon_n. \qquad \square$$

Definition 3.3. Let $\epsilon > 0$. A definable embedded submanifold M of K^n is **ϵ-flat** if for each $x, y \in M$ we have $\delta(TM_x, TM_y) < \epsilon$, where TM_x denotes the tangent space to M at x.

Lemma 3.4. *Let $A \subset K^n$ be a definable submanifold of dimension e and $\epsilon \in \mathbb{R}_{>0}$. Then there is a cell decomposition $A = \bigcup_{i=0}^{k} A_i$ of A such that for every i we have either $\dim(A_i) < \dim(A)$ or A_i is an ϵ-flat submanifold of K^n.*

Proof. Cover $\mathcal{G}_e(K^n)$ by a finite number of balls B_i of radius $\epsilon/2$; and consider the Gauss map $G : A \to \mathcal{G}_e(K^n)$ taking an element a of A to TA_a. Take a cell decomposition of K^e compatible with A and partitioning each $G^{-1}(B_i)$. Then the e-dimensional cells contained in A are ϵ-flat. ☐

Lemma 3.5. *Let $\epsilon \in \mathbb{Q}_{>0}$, and let $A \subset \mathring{K}^n$ be an open definable set. Then there are open, pairwise disjoint cells $A_1, \dots, A_p \subset A$ such that*

(i) $\dim(A - \bigcup A_i) < n$.

(ii) *For each i, there are definable, pairwise disjoint sets $B_1, \dots, B_k$ (with k depending on i) such that*

  (a) $k \leq 2n$;

  (b) *each B_j is a definable subset of ∂A_i and an ϵ-flat, $(n{-}1)$-dimensional, $\mathcal{C}^1$-submanifold of K^n;*

  (c) $\dim(\partial A_i - \bigcup_{j=1}^k B_j) < n - 1$.

Proof. By induction on n. The lemma is clear for $n = 1$. Assume that $n > 1$ and the lemma holds for smaller values of n.

Take a cell decomposition of $\overline{A}$ compatible with A into $\mathcal{C}^1$-cells. Let C be an open cell in this decomposition; it suffices to prove the lemma for C. Note that $C = (f, g)_X$, where X is an open cell in K^{n-1} and f, g are definable $\mathcal{C}^1$-functions on X. Take finite covers of $\Gamma(f)$ and $\Gamma(g)$ by open, definable sets U_i and V_j, respectively, such that each $U_i \cap \Gamma(f)$ and each $V_j \cap \Gamma(g)$ is ϵ-flat (to do this, take a finite cover of the Grassmannian by ϵ-balls and pull it back via the Gauss maps for $\Gamma(f)$ and $\Gamma(g)$). The collection of all sets $\pi(U_i) \cap \pi(V_j)$ is an open cover $\mathcal{O}$ of X. By the cell decomposition theorem, there is a $\mathcal{C}^1$-cell decomposition of X partitioning each set in $\mathcal{O}$. Let S be an open cell in this decomposition, and let $C_0 := (f, g)_S$. It suffices to prove the lemma for C_0. By the inductive hypothesis, we can find $A'_1, \dots, A'_p \subset S$ and $B'_1, \dots, B'_k \subset \partial A'_i$ satisfying the conditions (i) and (ii) above (with n replaced by $n - 1$). Define

$$A_i := (f, g)_{A'_i}, \qquad i = 1, \dots, p.$$

Then $\dim(C_0 - \bigcup_{i=1}^p A_i) < n$. For $j = 1, \dots, k$, the set $(B'_j \times K) \cap \partial A_i$ is definable. Take a $\mathcal{C}^1$-cell decomposition of this set, and let B_j be the union of the $(n - 1)$-dimensional cells in this decomposition (note that B_j may be empty). Then B_j is an ϵ-flat $\mathcal{C}^1$-submanifold of K^n and

$$\dim\big(((B'_j \times K) \cap \partial A_i) - B_j\big) < n - 1.$$

Define $B_{k+1} := \Gamma(f|A'_i)$ and $B_{k+2} := \Gamma(g|A'_i)$; by construction these are ϵ-flat. It is routine to see that $\partial A_i \subset B_{k+1} \cup B_{k+2} \cup (\partial A'_i \times K)$. Thus

$$\begin{aligned}\partial A_i - \bigcup_{j=1}^{k+2} B_j &\subset ((\partial A'_i \times K) \cap \partial A_i) - \bigcup_{j=1}^{k} B_j \\ &= (\bigcup_{j=1}^{k}((B'_j \times K) \cap \partial A_i) \cup E) - \bigcup_{j=1}^{k} B_j \\ &\subset \bigcup_{j=1}^{k}(((B'_j \times K) \cap \partial A_i) - B_j) \cup E,\end{aligned}$$

where E is a definable set with $\dim(E) < n-1$. Therefore $\dim(\partial A_i - \bigcup_{j=1}^{k+2} B_j) < n-1$. Since $k \leq 2(n-1)$, we get $k+2 \leq 2n$ and the lemma is proved. □

Definition 3.6. Let $U \subseteq K^n$ be open and let $f : U \to K^m$ be definable. Given $0 < M \in K$, we say that f is an M-function if $|\mathrm{D}f| \leq M$. We say that f has finite derivative if $|\mathrm{D}f|$ is finite.

Notice that, by ω-saturation of K, if f is definable and has finite derivative, then it is an M-function for some finite M.

Let $M \in K_{>0}$. An M-cell is a $\mathcal{C}^1$-cell where the $\mathcal{C}^1$ functions that define the cell are M-functions. More precisely:

Definition 3.7. Let $(i_1, \dots, i_m)$ be a sequence of zeros and ones, and $M \in K_{>0}$. An $(i_1, \dots, i_m)$-M-cell is a subset of K^m defined inductively as follows:

(i) A (0)-M-cell is a point $\{r\} \subset K$, a (1)-M-cell is an interval $(a, b) \subset K$, where $a,\ b \in K$.

(ii) An $(i_1, \dots, i_m, 0)$-M-cell is the graph $\Gamma(f)$ of a definable M-function $f : X \to K$ of class $\mathcal{C}^1$, where X is an $(i_1, \dots, i_m)$-M-cell; an $(i_1, \dots, i_m, 1)$-M-cell is a set

$$(f, g)_X := \{(x, r) \in X \times K : f(x) < r < g(x)\},$$

where X is an $(i_1, \dots, i_m)$-M-cell and $f, g : X \to K$ are definable M-functions of class $\mathcal{C}^1$ on X such that for all $x \in X$, $f(x) < g(x)$.

Theorem 3.8. *Let $A \subset \mathring{K}^n$ be definable. Then there are definable, pairwise disjoint sets A_i, $i = 1, \dots, s$, such that $A = \bigcup_i A_i$ and for each A_i, there is a change of coordinates $\sigma_i \in O_n(K)$ such that $\sigma_i(A_i)$ is an M_n-cell, where $M_n \in \mathbb{Q}_{>0}$ is a constant depending only on n.*

Proof. We will make use of the following fact:

Let $\epsilon \in [0, 1]$, $P \in \mathcal{G}_1(K^n)$, $X \in \mathcal{G}_k(K^n)$ and and $w \in X$ be a unit vector. Suppose $\delta(P, X) > \epsilon$. If $\pi_P(w) \geq 1/2$, where π_P is the orthogonal projection onto P, then

$$|\pi_P(w) - w| \geq |\pi_P(w) - \pi_X(\pi_P(w))| > |\pi_P(w)|\epsilon \geq 1/2\epsilon.$$

If $\pi_P(w) < 1/2$, then $|w| \leq |\pi_P(w)| + |\pi_p(w) - w| \leq 1/2 + |\pi_p(w) - w|$. In either case, we have

$$|\pi_P(w) - w| \geq \frac{1}{2}\epsilon. \tag{3}$$

We prove the theorem by induction on n; for $n = 1$ the theorem is clear. We assume that $n > 1$ and that the theorem holds for smaller values of n. We also proceed by induction on $d := \dim(A)$. It's clear for $d = 0$; so we assume that $d > 0$ and the theorem holds for definable bounded subsets B of K^n with $\dim(B) < d$.

Case I: $\dim(A) = n$. In this case A is an open, bounded, definable subset of K^n, so by using the inductive hypothesis and Lemma 3.5, we can reduce to the case where there are pairwise disjoint, definable $B_1, \ldots, B_k \subset \partial A$ such that $k \leq 2n$, $\dim(\partial A - \bigcup_{j=1}^k B_j) < n-1$ and each B_j is an ϵ_n-flat submanifold, where ϵ_n is as in Lemma 3.2. By Lemma 3.2, there is a hyperplane L such that for each B_j and all $x \in B_j$, we have $\delta(L^\perp, T_x B_j) > \epsilon_n$. Take a cell decomposition $\mathcal{B}$ of K^n, with respect to orthonormal coordinates in the L, $L^\perp$ axis, partitioning each B_j. Let

$$\mathcal{S} := \{C \in \mathcal{B} : \dim(C) = n-1, C \subset \textstyle\bigcup_{j=1}^k B_j\}$$

and note that $\dim(\partial A \setminus \bigcup_{C \in \mathcal{S}} C) < n - 1$. Furthermore,

$$\mathsf{BAD} := \{x \in A : \pi_L^{-1}(\pi_L(x)) \cap \partial A \not\subset \textstyle\bigcup_{c \in \mathcal{S}} C\}$$

has dimension smaller than n. Let $U_1, \ldots, U_l$ be the elements of $\{\pi_L(C) : C \in \mathcal{S}\}$. Then the set

$$\{x \in A : x \notin \pi_L^{-1}(\textstyle\bigcup_{i=1}^l U_i)\}$$

is contained in BAD, and therefore has dimension smaller than n.

By using the inductive hypothesis, we only need to find the required partition for each of the sets $A \cap \pi_L^{-1}(U_i)$, $i = 1, \ldots, l$. Fix $i \in \{1, \ldots, l\}$ and let $U := U_i$, $A' := A \cap \pi_L^{-1}(U)$. Take $C \in \mathcal{S}$ with $\pi_L(C) = U$. Then $C = \Gamma(\phi)$ for a definable $\mathcal{C}^1$-map $\phi : U \to L^\perp$ and for all $x \in C$,

$$T_x C = \{(v, \mathrm{D}\phi(v)) : v \in T_{\pi_L(x)} U\}.$$

Let $v \in T_{\pi_L(x)} U$ be a unit vector; since $\delta(L^\perp, T_x C) > \epsilon_n$ and $|(v, \mathrm{D}\phi(v))| = \sqrt{1 + |\mathrm{D}\phi(v)|^2}$, it follows from equation (3) that

$$\frac{1}{2}\epsilon_n \leq \frac{1}{\sqrt{1 + |\mathrm{D}\phi(v)|^2}} |\pi_{L^\perp}((v, \mathrm{D}\phi(v))) - (v, \mathrm{D}\phi(v))| = \frac{1}{\sqrt{1 + |\mathrm{D}\phi(v)|^2}} |v|.$$

Therefore,

$$|\mathrm{D}\phi(v)| \leq \sqrt{\frac{4}{\epsilon_n^2} - 1}.$$

Let $M_n \in \mathbb{Q}$ be bigger than $\max\left\{M_{n-1}, \sqrt{\frac{4}{\epsilon_n^2} - 1}\right\}$.

We have proved that for each $C_j \in \mathcal{S}$ with $\pi_L(C_j) = U$ there is a definable $\mathcal{C}^1$-map $\phi_j : U \to K$, such that $|\mathrm{D}\phi_j| < M_n$ and $C_j = \Gamma(\phi_j)$.

By the inductive hypothesis, there is a partition $\mathcal{P}$ of U such that each piece $P \in \mathcal{P}$ is a M_{n-1}-cell after a change of coordinates of L. We have

$$A' = \coprod_{\substack{P \in \mathcal{P} \\ (\phi_r, \phi_s)_P \subset A'}} (\phi_r, \phi_s)_P,$$

and $(\phi_r, \phi_s)_P$ is a M_n-cell after a coordinate change.

Case II: $\dim(A) < n$. In this case, by Lemma 3.4, we can partition A into cells which are ϵ_n-flat. Therefore we may assume that A is an ϵ_n-flat submanifold, where ϵ_n is as in Lemma 3.2. As in case I, there is a hyperplane L such that A is the graph of a function $f : U \to K$, $U \subset L$ and $|\mathrm{D}f| < M_n$. By the inductive hypothesis, we can partition U into M_{n-1}-cells. The graphs of f over the cells in this partition give the required partition of A. ☐

Definition 3.9. Let $A \subseteq K^n$ and $e \leq n$. A is basic e-rectifiable with bound M if, after a permutation of coordinates, A is the graph of an M-function $f : U \to K^{n-e}$, where $U \subset K^e$ is an open M-cell for some finite M.

Lemma 3.10. *Let $A \subset \mathring{K}^n$ be an M-cell of dimension e. Then, A is a basic e-rectifiable set, and the bound of A can be chosen depending only on M and n.*

Proof. We proceed by induction on n. If $n = 0$ or $n = 1$ the result is trivial, so assume $n \geq 2$. By definition, there exists an M-cell $B \subset \mathring{K}^{n-1}$ such that

(1) either $A = \Gamma(g)$ for some M-function $g : B \to \mathring{K}$, or

(2) $A = (g, h)_B$ for some M-functions $g, h : B \to \mathring{K}$, with $g < h$.

By inductive hypothesis, there exists an open L-cell $C \subset K^d$ (for some d and some $L \geq M$ depending only on M and on n), and an L-function $f : C \to K^{n-1-d}$, such that $B = \Gamma(f)$.

In case (1) $d = e$. Define $l : C \to K^{n-e}$ by $l(x) = \langle f(x), g(x, f(x))\rangle$. It is easy to see that l is an L'-function for some L' depending only on M and n, and that $A = \Gamma(l)$.

In case (2), $d = e-1$. Define $\tilde{g} := g \circ f$, $\tilde{h} := h \circ f$, and $\tilde{B} := (\tilde{g}, \tilde{h})_C$. Given $\langle \bar{x}, y \rangle \in \tilde{B}$, define $l(\bar{x}, y) := f(\bar{x})$. We have that $\tilde{B}$ is an open e-dimensional L-cell, $l : \tilde{B} \to K^{n-e}$ is an L-function, and $A = \Gamma(l)$. □

Corollary 3.11. *Let $A \subseteq K^n$ be definable of dimension at most e. Then there is a partition $A = \bigcup_{i=0}^{k} A_i$ such that $\dim(A_0) < e$ and A_i is a basic e-rectifiable set for $i > 0$. Moreover, the bounds of each A_i can be chosen to depend only on n (and not on A). We call $(A_0, \ldots, A_k)$ a basic e-rectifiable partition of A.*

Proof. Apply Theorem 3.8 and 3.10. □

Notice that a similar result has also been proved in [PW06, Theorem 2.3] (where they also take arbitrarily small bounds): however, in [PW06] they don't require that the functions parametrizing the set A are injective (which is essential for our later uses).

4 Whitney decomposition

The fact that the functions that define an M-cell are actually Lipschitz function follows from the following property of M-cells:

> Every pair of points x, y in an M-cell $C \subset K^n$ can be connected by a definable $\mathcal{C}^1$ curve $\gamma : [0,1] \to C$ with $|\gamma'(t)| < N|x - y|$, where N is a constant depending only on M and n which is finite if M is (Lemma 4.3 or [VR06] 3.10 & 3.11).

The same property implies that a N-function f on an M-cell is Lipschitz where the Lipschitz constant is finite if M and N are (Corollary 4.5). This last property will be needed for defining Hausdorff measure.

Remark 4.1. Let $U \subset \mathring{K}^n$ be open and definable, and $f : U \to \mathring{K}$ be an M-function (for some finite M). It is not true in general that f is L-Lipschitz for some finite L: this is the reason why we needed to prove Theorem 3.8.

Definition 4.2. Let $A \subset K^n$, $B \subset K^m$ be definable sets. Let $\lambda \subset A \times ([0,1] \times B) \subset K^n \times K^{1+m}$ be a definable set such that for every $x \in A$, the fiber over x

$$\lambda_x := \{y \in [0,1] \times B : \langle x, y \rangle \in \lambda\}$$

is a curve $\lambda_x : [0,1] \to B$. We view λ as describing the family of curves $\{\lambda_x\}_{x \in A}$. Such a family is a definable family of curves (in B, parametrized by A).

An L-cell is an L-Lipschitz cell if the functions that define the L-cell are L-Lipschitz.

Lemma 4.3. *Fix $L \in K_{>0}$ and $n \in \mathbb{N}_{>0}$. Then, there is a constant $K(n, L) \in K_{>0}$ depending only on n and L, that is finite if L is, such that for every L-Lipschitz cell $C \subset K^n$ there is a definable family of curves $\gamma \subset C^2 \times ([0,1] \times C)$ such that: For all $x, y \in C$, $\gamma_{x,y} : [0,1] \to C$ is a $\mathcal{C}^1$-curve with*

(i) $\gamma_{xy}(0) = x, \gamma_{xy}(1) = y$;

(ii) $|\gamma'_{xy}(t)| \leq K(n, L)|x - y|$, *for all* $t \in [0,1]$.

Proof. By induction on n. For $n = 1$ the lemma is clear. Take $n \geq 1$, and assume that the lemma holds for n. Let $C \subset K^{n+1}$ be an L-Lipschitz cell. Then $C = \Gamma(f)$ or $C = (g, h)_X$ for some L-Lipschitz cell $X \subset K^{n-1}$ and definable, $\mathcal{C}^1$, L-Lipschitz functions f, g, h with $g < h$, and $|\mathrm{D}f|, |\mathrm{D}g|, |\mathrm{D}h| \leq L$. By induction, there are a constant $k := K(n-1, L)$ and a definable family of $\mathcal{C}^1$-curves β in X with the required properties. Let $\pi_n : K^{n+1} \to K^n$ be the projection onto the first n coordinates.

If $C = \Gamma(f)$, we lift β to C via f: fix $x, y \in C$ and let $\gamma_{x,y}(t) := (\alpha(t), f(\alpha(t)))$, where for all $t \in [0,1]$ $\alpha(t) := \beta_{\pi_n(x),\pi_n(y)}(t)$. Then we have $|\gamma'_{xy}(t)| \leq (1 + L)k|x - y|$.

If $C = (g, h)_X$, we lift β as follows: Fix $x, y \in C$ and let $\alpha := \beta_{\pi_n(x),\pi_n(y)}$. Let $\pi : K^{n+1} \to K$ be the projection onto the last coordinate and take $u, v \in (0,1)$ with

$$\pi(x) = uh(\alpha(0)) + (1 - u)g(\alpha(0))$$
$$\pi(y) = vh(\alpha(1)) + (1 - v)g(\alpha(1)).$$

Let $l(t) := tv + (1-t)u$, for $t \in [0,1]$. We define $\gamma_{x,y}(t) := (\alpha(t), l(t)h(\alpha(t)) + (1 - l(t))g(\alpha(t)))$, and note that

$$|\gamma'_{xy}(t)| \leq k|x - y| + |(v - u)(h(\alpha(t)) - g(\alpha(t)))| + 2Lk|x - y|,$$

since $l(t), 1 - l(t)$ are between 0 and 1 and $|\mathrm{D}h(\alpha'(t))|, |\mathrm{D}g(\alpha'(t))| \leq L|\alpha'(t)|$. Let $f := h - g$. We want to bound $|(v - u)f(\alpha(t))|$, which equals

$$|\pi y - \pi x - v(f(\alpha(1)) - f(\alpha(t))) + u(f(\alpha(0)) - f(\alpha(t))) + g(\alpha(0)) - g(\alpha(1))|.$$

But

$$|f(\alpha(1)) - f(\alpha(t))| \leq L|\alpha(1) - \alpha(t)| = L|1 - t| \left| \frac{\alpha(1) - \alpha(t)}{1 - t} \right| \leq L|\alpha'(t_0)|$$

for some t_0 between t and 1. Similarly, $|f(\alpha(0)) - f(\alpha(t))| \leq L|\alpha'(t_1)|$, for some t_1 between t and 1. Since $u, v \in [0,1]$, we get

$$|(v-u)f(\alpha(t))| \leq |\pi y - \pi x| + 2Lk|x-y| + L|x-y|;$$

thus $|\gamma'_{xy}(t)| \leq K(n,L)|x-y|$ for some constant $K(n,L)$ depending only on n and L which is finite if L is. The collection of the curves γ_{xy} for $x, y \in C$ constitutes the required family of curves. □

Theorem 4.4. *Let $L > 0$, and let $C \subset K^n$ be an L-cell. Then C is a $k(n,L)$-Lipschitz cell, where $k(n,L)$ depends only on n and L, and is finite if L is.*

Proof. By induction on n; the theorem is clear for $n = 1$. Assume that $n > 1$ and that the theorem holds for $n-1$. Then $C = \Gamma(f)$ or $C = (g,h)_X$, where $X \subset K^{n-1}$ is a $k(n-1,L)$-Lipschitz cell and f, g, h are $\mathcal{C}^1$-functions on X such that $|\mathrm{D}f|, |\mathrm{D}g|, |\mathrm{D}h| \leq L$. We need to show that f, g, h are Lipschitz.

Since X is a k-Lipschitz cell, $k := k(n-1,L)$, it follows from Lemma 4.3 that there is a constant $K(n-1,k)$ such that whenever $x, y \in X$, there is a definable, $\mathcal{C}^1$-curve γ joining x and y with $|\gamma'(t)| \leq K(n-1,k)|x-y|$ for all $t \in [0,1]$. Let $g := f \circ \gamma$, and let $t_0 \in (0,1)$ be such that

$$|f(x) - f(y)| = |g'(t_0)| = |\mathrm{D}f(\gamma'(t_0))| \leq L|\gamma'(t_0)| \leq LK(n-1,k)|x-y|.$$

Thus f is $LK(n-1,k)$-Lipschitz. We set $k(n,L) := LK(n-1,k)$. □

Corollary 4.5. *Let C be an M-cell and f be a definable M-function. Then f is Lipschitz, and with finite Lipschitz constant if M is finite.*

Proof. By Theorem 4.4, C has a definable family of curves as in Lemma 4.3. The result therefore follows from the mean value theorem. □

Definition 4.6. A definable set $A \subset K^n$ satisfies the Whitney arc property if there is a constant $K \in \mathring{K}_{>0}$ such that for all $x, y \in A$ there is a definable curve $\gamma : [0,1] \to A$ with $\gamma(0) = x$, $\gamma(1) = y$ and $\mathrm{length}(\gamma) := \int_0^1 |\gamma'| \leq K|x-y|$.

Lemma 4.7. *Let $C \subset \mathring{K}^n$ be an M-cell, $M \in \mathring{K}$. Then, C satisfies the Whitney arc property.*

Proof. It follows from Theorem 4.4 and Lemma 4.3. □

Theorem 4.8. *Let $A \subset \mathring{K}^n$ be definable. Then, A can be partitioned into finitely many definable sets, each of them satisfying the Whitney arc property.*

Proof. This follows from Lemma 4.7, Theorem 3.8 and the fact that the Whitney arc property is invariant under an orthonormal change of coordinates. □

5 Hausdorff measure

For an introduction to geometric measure theory, and in particular to the Hausdorff measure, see [Morgan88].

Definition 5.1. Let $U \subseteq K^n$ be open and let $f : U \to \mathring{K}^m$ be a definable function. If $a \in U$, $e \leq n$ and M is the set of the $e \times e$ minors of $\mathrm{D}f(a)$ we define

$$J_e f(a) = \begin{cases} +\infty & \text{if } f \text{ is not differentiable at } a \text{ or } \mathrm{rank}(\mathrm{D}f(a)) > e, \\ \sqrt{\sum_{m \in M} m^2} & \text{otherwise;} \end{cases}$$

(cf. [Morgan88, §3.6]).

Notice that if $e = n = m$, then $J_n f = |\det(\mathrm{D}f)|$.

Definition 5.2. Let $U \subseteq \mathring{K}^e$ be an open M-cell for some $M \in \mathbb{N}$, and let $f : U \to \mathring{K}^m$ be a definable function with finite derivative. Let $F : U \to \mathring{K}^{m+e}$ be $F(x) := \langle x, f(x) \rangle$ and $C := \Gamma(f) = F(U)$ (notice that C has bounded diameter). We define

$$\mathcal{H}^e(C) := \int_U J_e F \,\mathrm{d}\mathcal{L}^e.$$

Lemma 5.3. *If $C \subseteq \mathring{K}^n$ is basic e-rectifiable, then $\mathcal{H}^e(C) = \mathcal{H}^e_{\mathbb{R}}(\mathrm{st}(C))$, where $\mathcal{H}^e_{\mathbb{R}}$ is the e-dimensional Hausdorff measure on $\mathbb{R}^n$.*

Proof. Let $A \subset \mathring{K}^e$ and $f : A \to \mathring{K}^{n-e}$ be as in Definition 3.9, and $F : A \to \mathring{K}^n$ as in Definition 5.2. Let $B := \mathrm{st}(A)$. Then, using the real Area formula [Morgan88],

$$\int_A J_e F \,\mathrm{d}\mathcal{L}^e = \int_B J_e(\overline{F}) \,\mathrm{d}\mathcal{L}^e_{\mathbb{R}} = \mathcal{H}^e_{\mathbb{R}}(\overline{F}(B)) = \mathcal{H}^e_{\mathbb{R}}(\mathrm{st}(C)).$$

□

Definition 5.4. Let $A \subseteq \mathring{K}^n$ be definable of dimension at most e, and $(A_0, \dots, A_k)$ be a basic e-rectifiable partition of A. Define

$$\mathcal{H}^e(A) := \sum_{i=1}^k \mathcal{H}^e(A_i),$$

where $\mathcal{H}^e(A_i)$ is defined using 5.2.

Lemma 5.5. *If A is as in the above definition, then $\mathcal{H}^e(A)$ does not depend on the choice of the basic e-rectifiable partition $(A_0, \dots, A_k)$.*

Proof. It suffices to prove the following: if C is a basic e-rectifiable set and $(A_0,\dots,A_k)$ is a basic e-rectifiable partition of C, then $\mathcal{H}^e(C)=\sum_{i=1}^k \mathcal{H}^e(A_i)$, where $\mathcal{H}^e(C)$ and $\mathcal{H}^e(A_i)$ are defined using 5.2. For every $i=1,\dots,n$ let U and V_i be M-cells, $f: U \to K^{n-e}$ and $g_i : V_i \to K^{n-e}$ be definable functions with finite derivative, σ_i be a permutation of variables of K^n, $F: K^e \to K^n$ defined by $F(x) := (x, f(x))$, and $G_i : K^e \to K^n$ defined by $G(x) = \sigma_i(x, g_i(x))$ such that $C = F(U)$ and $A_i = G_i(V_i)$. Define $U_i := F^{-1}(A_i) \subseteq U$, and $H_i := G_i^{-1} \circ F : U_i \to V_i$. Notice that each H_i is a bi-Lipschitz bijection, that U is the disjoint union of the U_i, and that $\dim(U_0) < e$. Hence,

$$\mathcal{H}^e(C) = \int_U J_e F \,\mathrm{d}\mathcal{L}^e = \sum_{i=1}^n \int_{U_i} J_e F \,\mathrm{d}\mathcal{L}^e = \sum_{i=1}^n \int_{U_i} J_e(G_i \circ H_i)\,\mathrm{d}\mathcal{L}^e =$$

$$= \sum_{i=1}^n \int_{U_i} (J_e(G_i) \circ H_i) \cdot |\det(\mathrm{D}H_i)| \mathrm{d}\mathcal{L}^e = \sum_{i=1}^n \int_{V_i} J_e G_i \mathrm{d}\mathcal{L}^e = \sum_{i=1}^n \mathcal{H}^e(A_i),$$

where we used Lemma 2.9, the fact that each σ_i is a linear function with determinant ± 1, and that $J_e(G \circ H) = (J_e(G) \circ H) \cdot |\det(\mathrm{D}H)|$. □

Lemma 5.6. *$\mathcal{H}^e$ does not depend on n. That is, let $m \geq n$, and $A \subset \mathring{K}^n$ definable, and $\psi : K^n \to K^m$ be the embedding $x \mapsto (x, 0)$. Then, $\mathcal{H}^e(A) = \mathcal{H}^e(\psi(A))$.*

Proof. Obvious from the definition and Lemma 5.5. □

Notice that $\mathcal{H}^0(C)$ is the cardinality of C.

It is clear that $\mathcal{H}^e$ can be extended to the σ-ring generated by the definable subsets of K^n of finite diameter and dimension at most e; we will also denote the completion of this extension by $\mathcal{H}^e$.

Lemma 5.7. *$\mathcal{H}^e$ is a measure on the σ-ring generated by the definable subsets of K^n of bounded diameter and dimension at most e.*

Proof. Since K is $\aleph_1$-saturated, it suffices to show that, for every A and B disjoint definable subsets of K^n of finite diameter and dimension at most e, $\mathcal{H}^e(A \cup B) = \mathcal{H}^e(A) + \mathcal{H}^e(B)$. But this follows immediately from Lemma 5.5. □

Example 5.8. In Lemma 5.3, the assumption that C is basic e-rectifiable is necessary. For instance, take $\epsilon > 0$ infinitesimal, and X be the following subset of K^2

$$X := \big([0,1] \times \{0\}\big) \cup \{\langle x, y \rangle : 0 \leq x \leq 1 \;\&\; y = \epsilon x\}.$$

Then, $\mathrm{st}(X) = [0,1] \times \{0\}$, and thus $\mathcal{H}^1(X) = 2$, while $\mathcal{H}^1_{\mathbb{R}}(\mathrm{st}(X)) = 1$. This is the source of complication in the theory, and one of the reasons why we had to wait until this section to introduce $\mathcal{H}^e$.

6 Cauchy-Crofton formula

Give $e \le n$, define

$$\beta := \Gamma\left(\tfrac{e+1}{2}\right)\Gamma\left(\tfrac{n-e+1}{2}\right)\Gamma\left(\tfrac{n+1}{2}\right)^{-1}\pi^{-1/2}.$$

Definition 6.1. Let $\mathcal{AG}_e(K^n)$ be the Grassmannian of affine e-dimensional subspaces of K^n and let $\mathcal{AG}_e(\mathbb{R}^n)$ be the Grassmannian of affine e-dimensional subspaces of $\mathbb{R}^n$. Fix an embedding of $\mathcal{AG}_e(\mathbb{R}^n)$ into some $\mathbb{R}^m$, such that $\mathcal{AG}_e(\mathbb{R}^n)$ is a $\emptyset$-semialgebraic closed submanifold of $\mathbb{R}^m$, and the restriction to $\mathcal{AG}_e(\mathbb{R}^n)$ of the $\dim(\mathcal{AG}_e(\mathbb{R}^n))$-dimensional Hausdorff measure coincides with the Haar measure on $\mathcal{AG}_e(\mathbb{R}^n)$.

Definition 6.2. Given $A \subseteq K^n$ and $E \in \mathcal{AG}_{n-e}(K^n)$, let $f_A(E) := \#(A \cap E)$.

Theorem 6.3 (Cauchy-Crofton Formula)**.** *Let $A \subseteq \mathring{K}^n$ be definable of dimension e. Then,*

$$\mathcal{H}^e(A) = \frac{1}{\beta} \int\limits_{\mathcal{AG}_{n-e}(K^n)} f_A \, \mathrm{d}\mathcal{L}^{\mathcal{AG}_{n-e}(K^n)}.$$

We prove the theorem by reducing it to the known case of $K = \mathbb{R}$. This is done by showing that $\#(A \cap E)$ equals $\#(\mathrm{st}\, A \cap \mathrm{st}\, E)$ almost everywhere.

Definition 6.4. Let $f : U \to \mathring{K}^m$ be definable, with $U \subset \mathring{K}^n$ open. Let $E \subset \mathbb{R}^n$ and $\overline{f}$ be as in Lemma 2.10. We say that $b \in \mathbb{R}^n$ is an S-regular point of $\overline{f}$ if

i) $b \in \mathrm{st}(U) \setminus \overline{E}$;

ii) b is a regular point of $\overline{f}$.

Otherwise, we say that b is an S-singular point and $\overline{f}(b)$ is an S-singular value of $\overline{f}$. If $c \in \mathbb{R}^m$ is not an S-singular value, we say that c is an S-regular value of $\overline{f}$.

Remark 6.5. Let S be the set of S-regular points of $\overline{f}$. Then, S is open and definable in $\mathbb{R}_K$.

Lemma 6.6 (Morse-Sard)**.** *Assume that* $m \geq n$*. Then, the set of* S*-singular values of* $\overline{f}$ *is* $\mathcal{L}^m_{\mathbb{R}}$*-negligible,*

Proof. By Lemma 2.10, E is negligible; since E is also $\mathbb{R}_K$-definable, it has empty interior and therefore $\dim(E) < n$. Since $m \geq n$, it follows that $\overline{f}(E)$ is negligible. The set of S-singular values of $\overline{f}$ is the union of $\overline{f}(E)$ and the set of singular values of $\overline{f}$; it is therefore negligible. □

Lemma 6.7 (Implicit Function)**.** *Assume that* $m = n$*. Let* $b \in \mathbb{R}^n$*. If* b *is an* S*-regular point of* $\overline{f}$ *then, for every* $y \in \mathrm{st}^{-1}(\overline{f}(b))$ *there exists a unique* $x \in \mathrm{st}^{-1}(b)$ *such that* $f(x) = y$*.*

Proof. Choose $x_0 \in \mathrm{st}^{-1}(b)$. Let $A := (\mathrm{D}f(x_0))^{-1}$. Since b is a regular point of $\overline{f}$, $\|A\|$ is finite. Thus we can choose $r, \rho \in \mathbb{Q}_{>0}$ such that $B := \overline{B(b;\rho)}$ is contained in the set of S-regular points of $\overline{f}$, and

$$\|\mathrm{D}\overline{f}(b') - \mathrm{D}\overline{f}(b)\| < \frac{1}{2n\|A\|}, \quad \text{for every } b' \in B$$
$$r \leq \frac{\rho}{2\|A\|}.$$

Moreover, we can pick ρ such that $B' := \overline{B(x_0;\rho)} \subset U$. Given $y \in K^n$ such that $|y - f(x_0)| < r$, consider the mapping

$$T_y : B' \to K^n$$
$$T_y(x) := x + A \cdot (y - f(x)).$$

T_y is definable and Lipschitz, with Lipschitz constant $1/2$. Therefore, for every $y \in B(f(x_0); r)$ there exists a unique $x \in B'$ such that $T_y(x) = x$. Thus, there is a unique $x \in B$ with $f(x) = y$. It remains to show that, given $y \in \mathrm{st}^{-1}(\overline{f}(b))$ and $x \in B'$ such that $f(x) = y$, we have $x \in \mathrm{st}^{-1}(b)$. We can verify that

$$\overline{T}_y : B \to B$$
$$\overline{T}_y(b') = b' + (\mathrm{D}\overline{f}(b))^{-1} \cdot (\overline{f}(b) - \overline{f}(b'))$$

is also a contraction, and therefore it has a unique fixed point, namely b. Since $\overline{T}_y(\mathrm{st}(x)) = \mathrm{st}(x)$, we must have $\mathrm{st}(x) = b$. □

Remark 6.8. Let $U \subset \mathring{K}^m$. If $f : U \to \mathring{K}^n$ is definable and M-Lipschitz (for some finite M), $n \geq m$ and E is $\mathcal{L}^m_{\mathbb{R}}$-negligible, then the set $f(\mathrm{st}^{-1}(E))$ is $\mathcal{L}^n$-negligible.

Proof. We can cover E with a polyrectangle Y whose measure is an arbitrarily small rational number λ and such that Y covers $\mathrm{st}^{-1}(E)$. Since $f(Y)$ has measure at most $CM^n\lambda$ (C depends only on m and n) the result follows. □

Lemma 6.9. *Let $A \subseteq \mathring{K}^n$ be a basic e-rectifiable set of dimension e. Consider $V := K^e$ as embedded in K^n via the map $x \mapsto \langle x, 0\rangle$. Identify each $p \in V$ with the $(n-e)$-dimensional affine space which is orthogonal to V and intersects V in p. Then, for almost every $p \in V$, we have $\#(p \cap A) = \#(\mathrm{st}(p) \cap \mathrm{st}(A))$.*

Proof. Let $\pi : K^n \to V$ be the orthogonal projection. Let $U \subset \mathring{K}^e$ be an open M-cell and $f : U \to K^{n-e}$ be a definable M-function (M finite) such that $A = \Gamma(f)$. Let $F(x) := \langle x, f(x)\rangle$. Let $h := \pi \circ F : U \to V$, and consider $\overline{h} : C \to \mathrm{st}(V)$, $C \subset \mathrm{st}(U)$ as in Lemma 2.10. For almost every $p \in V$, $\#(p \cap A) = \#(h^{-1}(p))$, and $\#(\mathrm{st}\, p \cap stA) = \#(\overline{h}^{-1}(\mathrm{st}\, p))$ because $F : U \to A$ and $\overline{F} : C \to \mathrm{Im}(\overline{F})$ are bijections. Thus, it suffices to prove that, for almost every $p \in V$, $\#(h^{-1}(p)) = \#(\overline{h}^{-1}(\mathrm{st}\, p))$. Let E be as in Lemma 2.10. By Remark 6.8, $h(\mathrm{st}^{-1}(E))$ is $\mathcal{L}^e$-negligible. Let S be the set of S-singular values of $\overline{h}$, by Lemma 6.6, S is negligible.

Let $p \in V \setminus (\mathrm{st}^{-1}(S) \cup h(\mathrm{st}^{-1}(E))$. Then for every x in $h^{-1}(p)$, $\mathrm{st}(x)$ is an S-regular point of $\overline{h}$, and therefore Lemma 6.7 implies $\#(h^{-1}(p)) = \#(\overline{h}^{-1}(\mathrm{st}\, p))$. ☐

Notice that the above lemma does not hold if A is only definable, instead of basic e-rectifiable.

Proof of Theorem 6.3. By Corollary 3.11, w.l.o.g. A is basic e-rectifiable. Let $B := \mathrm{st}(A)$, and $f_B(F) := \#(B \cap F)$, for every $F \in \mathcal{AG}_e(\mathbb{R}^n)$. By Lemma 6.9,

$$\int\limits_{\mathcal{AG}_{n-e}(K^n)} f_A \,\mathrm{d}\mathcal{L}^{\mathcal{AG}_{n-e}(K^n)} = \int\limits_{\mathcal{AG}_{n-e}(\mathbb{R}^n)} f_B \,\mathrm{d}\mathcal{L}^{\mathcal{AG}_{n-e}(\mathbb{R}^n)}.$$

By the usual Cauchy-Crofton formula [Morgan88, 3.16], the right-hand side in the above identity is equal to $\mathcal{H}^e_{\mathbb{R}}(B) = \mathcal{H}^e(A)$, where we applied Lemma 5.3. ☐

7 Further properties of Hausdorff measure and the Co-area formula

Theorem 7.1. *Let $e \le n$ and $C \subseteq K^n$ be bounded and definable of dimension at most e.*

1. *$\mathcal{H}^e$ is invariant under isometries.*
2. *For every $r \in \mathring{K}$, $\mathcal{H}^e(rC) = \mathrm{st}(r)^e \mathcal{H}^e(C)$.*

3. *If C is $\emptyset$-semialgebraic, then $\mathcal{H}^e(C) = \mathcal{H}^e(C_{\mathbb{R}}) = \mathcal{H}^e(\mathrm{st}(C))$.*

4. *if $\dim(C) < e$, then $\mathcal{H}^e(C) = 0$; the converse is not true.*

5. *$\mathcal{H}^e(C) < +\infty$.*

6. *If $\big(C(r)\big)_{r\in K^d}$ is a definable family of bounded subsets of K^n, then there exists a natural number M such that $\mathcal{H}^n(C(r)) < M$ for every $r \in K^d$.*

7. *If K' is either an elementary extension or an o-minimal expansion of K, then $\mathcal{H}^e(C_{K'}) = \mathcal{H}^e(C)$.*

8. *If $n = e$, then $\mathcal{H}^e(C) = \mathcal{L}^n(C)$.*

9. *If C is a subset of an e-dimensional affine space E, then $\mathcal{H}^e(C) = \mathcal{L}^E(C)$.*

Proof.

(1) Use the Cauchy-Crofton formula.

(2), (4) and (7) Apply the definition of $\mathcal{H}^e$ and Lemma 5.5.

(3) Apply Corollary 3.11 to $C_{\mathbb{R}}$ and use Lemma 5.3.

(5) and (6) Apply the Cauchy-Crofton formula: see [Dries03].

(8) Apply Lemma 5.3.

(9) Since $\mathcal{H}^e$ is invariant under isometries, w.l.o.g. E is the coordinate space K^e. By Lemma 5.6, the measure $\mathcal{H}^e$ inside K^n is equal to the measure $\mathcal{H}^e$ inside K^e, and the latter is equal to $\mathcal{L}^e$. The conclusion follows from Remark 2.1. ◻

The following theorem is the adaption to o-minimal structures of the Co-area formula, a well-known generalization of Fubini's theorem. Let $D := [0,1] \subset K$.

Theorem 7.2 (Co-area Formula)**.** *Let $A \subset D^m$ be definable, and $f : D^m \to D^n$ be a definable Lipschitz function, with $m \geq n$. Then, $J_n f$ is $\mathcal{L}^m_K$-integrable, and*

$$\int_A J_n f \, \mathrm{d}\mathcal{L}^m = \int_{D^n} \mathcal{H}^{m-n}(A \cap f^{-1}(y)) \, \mathrm{d}\mathcal{L}^n(y).$$

Sketch of Proof. W.l.o.g., A is an open subset of D^m. By Lemma 6.6, w.l.o.g. all points of A are S-regular for $\overline{f}$. Apply the real co-area formula [Morgan88] to $g := \overline{f}$ and $B := \mathrm{st}(A)$, and obtain

$$\int_A J_n f \,\mathrm{d}\mathcal{L}^m = \int_B J_n g \,\mathrm{d}\mathcal{L}^m_{\mathbb{R}} = \int_{D^n_{\mathbb{R}}} \mathcal{H}^{m-n}_{\mathbb{R}}(B \cap g^{-1}(z)) \,\mathrm{d}\mathcal{L}^n_{\mathbb{R}}(z).$$

By the Implicit Function Theorem and Lemma 5.3, for almost every $y \in D^n_{\mathbb{R}}$, we have

$$\mathcal{H}^{m-n}(A \cap f^{-1}(y)) = \mathcal{H}^{m-n}_{\mathbb{R}}(B \cap g^{-1}(\mathrm{st}\, y)).$$ □

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