Cell structure for the Yokonuma-Hecke algebra and related algebras

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Introduction

In this thesis, we study the representation theory of the Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}$ in type A and of the related Aicardi-Juyumaya algebra $\mathcal{E}_n$ of braids and ties. In the past few years, quite a few papers have been dedicated to the study of both algebras.

The Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}$ was first introduced in the sixties by Yokonuma [**44**] for general types as a generalization of the Iwahori-Hecke algebra $\mathcal{H}_n$, but the recent activity on $\mathcal{Y}_{r,n}$ was initiated by Juyumaya who in [**27**] gave a new presentation of $\mathcal{Y}_{r,n}$. It is a deformation of the group algebra of the wreath product $C_r \wr \mathfrak{S}_n$ of the cyclic group of order r, C_r, and the symmetric group $\mathfrak{S}_n$. On the other hand, it is quite different from the more familiar deformation of $C_r \wr \mathfrak{S}_n$, the Ariki-Koike algebra $\widetilde{\mathcal{H}}_{r,n}$. For example, the usual Iwahori-Hecke algebra $\mathcal{H}_n$ of type A appears canonically as a quotient of $\mathcal{Y}_{r,n}$, whereas it appears canonically as subalgebra of $\widetilde{\mathcal{H}}_{r,n}$.

Much of the impetus to the recent development on $\mathcal{Y}_{r,n}$ comes from knot theory. In the papers [**9**], [**10**], [**26**] and [**28**] a Markov trace on $\mathcal{Y}_{r,n}$ and its associated knot invariant Θ is studied.

The Aicardi-Juyumaya algebra $\mathcal{E}_n$ of braids and ties, along with its diagram calculus, was introduced in [**1**] and [**25**] via a presentation derived from the presentation of $\mathcal{Y}_{r,n}$. The algebra $\mathcal{E}_n$ is also related to knot theory. Indeed, Aicardi and Juyumaya constructed in [**2**] a Markov trace on $\mathcal{E}_n$, which gave rise to a three parameter knot invariant Δ. There seems to be no simple relation between Θ and Δ.

A main aim of this thesis is to show that $\mathcal{Y}_{r,n}$ and $\mathcal{E}_n$ are cellular algebras in the sense of Graham and Lehrer, [**16**]. On the way we give a concrete isomorphism between $\mathcal{Y}_{r,n}$ and Shoji's modified Ariki-Koike algebra $\mathcal{H}_{r,n}$. This gives two new proof of a result of Lusztig [**31**] and Jacon-Poulain d'Andecy [**21**], showing that $\mathcal{Y}_{r,n}$ is in fact a sum of matrix algebra over Iwahori-Hecke algebras of type A.

For the parameter $q = 1$, it was shown in Banjo's work [**4**] that the algebra $\mathcal{E}_n$ is a special case of P. Martin's ramified partition algebras. Moreover, Marin showed in [**32**] that $\mathcal{E}_n$ for $q = 1$ is isomorphic to a sum of matrix algebras over a certain wreath product algebra, in the spirit of Lusztig's and Jacon-Poulain d'Andecy's Theorem. He raised the question whether this result could be proved for general parameters. As an application of our cellular basis for $\mathcal{E}_n$ we do obtain such a structure Theorem for $\mathcal{E}_n$, thus answering in the positive Marin's question. Furthermore, we construct a cellular basis for the natural Temperley-lieb quotient of $\mathcal{E}_n$ defined by Juyumaya in [**24**].

Recently it was shown in [**9**] and [**38**] that the Yokonuma-Hecke algebra invariant Θ can be described via a formula involving the HOMFLYPT-polynomial and the linking number. In particular, when applied to classical knots, Θ and the HOMFLYPT-polynomial coincide (this was already known for some time). Given our results on $\mathcal{E}_n$ it would be interesting to investigate whether a similar result would hold for Δ.

The structure of this thesis is as follows. In Chapter 1 we recall some basic combinatorial objects related to the symmetric group and we also fix some notation. In Section 2 we define the two combinatorial object most important in the construction of our cellular basis for $\mathcal{Y}_{r,n}$ and $\mathcal{E}_n$; the multipartitions (tuple of partitions) and the set partitions of a number. In Section 3 we recall briefly the definition from [**16**] of a cellular algebra. Furthermore we give two examples which were the ones that inspired our main construction.

In Chapter 2 we study the theory of representations of the Yokonuma-Hecke algebra. In Section 1 we introduce the main objects of study for this chapter. The second part, Section 2, contains the construction of a faithful tensor space module $V^{\otimes n}$ for $\mathcal{Y}_{r,n}$. The construction of $V^{\otimes n}$ is a generalization of the $\mathcal{E}_n$-module structure on $V^{\otimes n}$ that was defined in [**39**] and it allows us to conclude that $\mathcal{E}_n$ is a subalgebra of $\mathcal{Y}_{r,n}$ for $r \geq n$, and for *any* specialization of the ground ring. The tensor space module $V^{\otimes n}$ is also related to the strange Ariki-Terasoma-Yamada action, [**3**] and [**40**], of the Ariki-Koike algebra on $V^{\otimes n}$, and thereby to the action of Shoji's modified Ariki-Koike algebra $\mathcal{H}_{r,n}$ on $V^{\otimes n}$, [**42**]. A speculating remark concerning this last point was made in [**39**], but the appearance of Vandermonde determinants in the proof of the faithfulness of the action of $\mathcal{Y}_{r,n}$ in $V^{\otimes n}$ makes the remark much more precise. The defining relations of the modified Ariki-Koike algebra also involve Vandermonde determinants and from this we obtain the proof of the isomorphism $\mathcal{Y}_{r,n} \cong \mathcal{H}_{r,n}$ by viewing both algebras as subalgebras of $\mathrm{End}(V^{\otimes n})$. Via this, we get a new proof of Lusztig's and Jacon-Poulain d'Andecy's isomorphism Theorem for $\mathcal{Y}_{r,n}$, since it is in fact equivalent to a similar isomorphism Theorem for $\mathcal{H}_{r,n}$, obtained independently by Sawada-Shoji and Hu-Stoll.

The third part of this chapter, Section 3, contains the proof that $\mathcal{Y}_{r,n}$ is a cellular algebra in the sense of Graham-Lehrer, via a concrete combinatorial construction of a cellular basis for it, generalizing Murphy's standard basis for the Iwahori-Hecke algebra of type A. The fact that $\mathcal{Y}_{r,n}$ is cellular could also have been deduced from the isomorphism $\mathcal{Y}_{r,n} \cong \mathcal{H}_{r,n}$ and from the fact that $\mathcal{H}_{r,n}$ is cellular, as was shown by Sawada and Shoji in [**41**]. Still, the usefulness of cellularity depends to a high degree on having a concrete cellular basis in which to perform calculations, rather than knowing the mere existence of such a basis, and our construction should be seen in this light.

Cellularity is a particularly strong language for the study of modular, that is non-semisimple representation theory, which occurs in our situation when the parameter q is specialized to a root of unity. But here our applications go in a different direction and depend on a nice compatibility property of our cellular basis with respect to a natural subalgebra of $\mathcal{Y}_{r,n}$. We get from this that the elements $m_{\mathfrak{s}\mathfrak{s}}$ of the cellular basis for $\mathcal{Y}_{r,n}$, given by one-column standard multitableaux $\mathfrak{s}$, correspond to certain idempotents that appear in Lusztig's presentation of $\mathcal{Y}_{r,n}$ in [**30**] and [**31**]. Using the faithfulness of the tensor space module $V^{\otimes n}$ for $\mathcal{Y}_{r,n}$ we get via this Lusztig's idempotent presentation of $\mathcal{Y}_{r,n}$. Thus the third part of this chapter depends logically on the second part. In the last part of this section we give another application of our cellular basis for $\mathcal{Y}_{r,n}$. We provide another proof of the fact that $\mathcal{Y}_{r,n}$ can be decomposed into a direct sum of matrix algebras by giving an explicit isomorphism between certain subalgebras of $\mathcal{Y}_{r,n}$ and certain matrix algebras over tensorial products of Hecke algebras. Moreover, this isomorphism preserves the cellular structure of these algebras since it sends cellular basis to cellular basis.

In Section 4 we treat the Jucys-Murphy's elements for $\mathcal{Y}_{r,n}$. They were already introduced and studied by Chlouveraki and Poulain d'Andecy in [**8**], but here we show that they are JM-elements in the abstract sense defined by Mathas, with respect to the cell structure that we found.

In Chapter 3, we study the representation theory of the algebra of braids and ties. In Section 1, we recall the definition of the algebra of braids and ties in terms of generators and relations and we use the tensor representation of $\mathcal{Y}_{r,n}$ constructed in the above chapter for to prove that $\mathcal{E}_n$ can be see as a subalgebra of $\mathcal{Y}_{r,n}$ which is not completely obvious. In Section 2, we use the Möebius function for to construct a complete set of central orthogonal idempotents $\{\mathbb{E}_\alpha | \alpha \in \mathcal{P}ar_n\} \subseteq \mathcal{E}_n$. From the general theory we have as an immediate consequence the following decomposition:

$$\mathcal{E}_n(q) = \bigoplus_{\alpha \in \mathcal{P}ar_n} \mathbb{E}_\alpha \mathcal{E}_n(q)$$

where $\mathbb{E}_\alpha \mathcal{E}_n(q)$ are subalgebras of $\mathcal{E}_n$.

In the third part of this chapter we construct a cellular basis for $\mathcal{E}_n$. This construction does not depend logically on the results of Chapter 2, but is still strongly motivated by them. The generic representation theory of $\mathcal{E}_n$ was already studied in [**39**] and was shown to be a blend of the symmetric group and the Hecke algebra representation theories and this is reflected in the cellular basis. The cellular basis is also here a variation of Murphy's standard basis but the details of the construction are substantially more involved than in the $\mathcal{Y}_{r,n}$-case.

In the last section we provide an application of our cellular basis. We show that $\mathcal{E}_n$ is isomorphic to a direct sum of matrix algebras over certain wreath product algebras $\mathcal{H}_\alpha^{wr}$, depending on a partition α. An essential ingredient in the proof of this result is a compatibility property of our cellular basis for $\mathcal{E}_n$ with respect to these subalgebras. It appears to be a key feature of Murphy's standard basis and its generalizations that they carry compatibility properties of this kind, see for example [**19**], [**12**] and [**13**], and thus our work can be viewed as a manifestation of this phenomenon.

CHAPTER 1

Preliminaries

In this chapter we set up the fundamental notation and introduce the objects we wish to investigate.

Throughout the thesis we fix the rings $R := \mathbb{Z}[q, q^{-1}, \xi, r^{-1}, \Delta^{-1}]$ and $S := \mathbb{Z}[q, q^{-1}]$, where q is an indeterminate, r is a positive integer, $\xi := e^{2\pi i/r} \in \mathbb{C}$ and Δ is the Vandermonde determinant $\Delta := \prod_{0 \le i < j \le r-1} (\xi^i - \xi^j)$.

We shall need the quantum integers $[m]_q$ defined for $m \in \mathbb{Z}$ by $[m]_q := \dfrac{q^{2m}-1}{q^2-1}$ if $q \neq 1$ and $[m]_q := m$ if $q = 1$.

1. The symmetric group

The symmetric group on n letters, $\mathfrak{S}_n$, is the group consisting of all bijections of the set $\{1, 2, \dots, n\}$ endowed with the operation of composition of functions. The elements of $\mathfrak{S}_n$ are called *permutations* and we choose the convention that they act on $\mathbf{n} := \{1, 2 \dots, n\}$ on the right.

Let $\Sigma_n := \{s_1, \dots, s_{n-1}\}$ be the set of simple transpositions in $\mathfrak{S}_n$, that is $s_i = (i, i+1)$ in cycle notation. Then $\mathfrak{S}_n$ can also be defined as the Coxeter group on Σ_n subject to the relations

$$s_i s_j = s_j s_i \qquad \text{for } |i-j| > 1 \tag{1.1}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad \text{for } i = 1, 2, \dots, n-2 \tag{1.2}$$

$$s_i^2 = 1 \qquad \text{for } i = 1, 2, \dots, n-1. \tag{1.3}$$

In particular, for each $w \in \mathfrak{S}_n$ there exist $i_j \in \{1, 2, \dots, n-1\}$ such that $w = s_{i_1} s_{i_2} \cdots s_{i_k}$. If k in minimal we say that w has length k and we write $\ell(w) = k$. Furthermore, in this case we say that $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a *reduced* expression for w. For example, $s_2 s_3 s_4 s_3 s_2 s_1 s_5 s_7 s_6 s_7$ is a reduced expression for the permutation

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 8 & 3 & 4 & 1 & 5 & 7 & 6 \end{pmatrix} \in \mathfrak{S}_8.$$

Then we have that $\ell(w) = 10$. Note that in general an element of $\mathfrak{S}_n$ will have many reduced expressions. For instance, by using the relations 1.1,1.2 and 1.3 on the above example, we have that $s_4 s_3 s_2 s_3 s_4 s_1 s_5 s_7 s_6 s_7$ is also a reduced expression for w.

The *word form* of $w \in \mathfrak{S}_n$ is obtained by listing from left to right the elements iw in increasing order with respect to i. For example, the word form 534216 represents the bijection $1 \to 5$, $2 \to 3$, $3 \to 4$, $4 \to 2$, $5 \to 1$ and $6 \to 6$. Denote by $\mathfrak{S}_n^W$ the set of elements of $\mathfrak{S}_n$, represented this way.

For $w \in \mathfrak{S}_n$ we define the *inversion number* of w as follows

$$\mathrm{inv}(w) := \mathrm{card}\{(i,j) | i < j, iw > jw\}.$$

Since $\mathfrak{S}_n$ acts on the right on $\mathbf{n}$, it also acts on the right on $\mathfrak{S}_n^W$. There is also an action of $\mathfrak{S}_n$ on $\mathfrak{S}_n^W$ given by interchanging the positions of the numbers in each word. To distinguish the two actions we consider the last action to be a left action. For example

$$s_2 s_1 \cdot 5763124 = 7653124 \;\text{ whereas }\; 5763124 \cdot s_1 s_2 = 5762314.$$

For all $w \in \mathfrak{S}_n$ we have

$$\mathrm{inv}(s_i w) = \begin{cases} \mathrm{inv}(w) + 1 & \text{if } iw < (i+1)w \\ \mathrm{inv}(w) - 1 & \text{if } iw > (i+1)w. \end{cases} \tag{1.4}$$

from which it follows that $l(w) = \mathrm{inv}(w)$.

Letting $long \in \mathfrak{S}_n$ be the element corresponding to $n(n-1)\cdots 21 \in \mathfrak{S}_n^W$ we get that $\mathrm{inv}(long) = l(long) = n(n-1)/2$ and for $w \in \mathfrak{S}_n$ we get that

$$\mathrm{inv}(w \cdot long) = \mathrm{inv}(long \cdot w) = n(n-1)/2 - \ell(w). \tag{1.5}$$

If $w = s_{i_k} \cdots s_{i_2} s_{i_1}$ is a reduced expression and if $v := s_{i_k} w$, then the word from for $w \cdot long$ is obtained from the word form for $v \cdot long$ by acting on the left with s_{i_k}. But from the above, we then conclude that the number in $v \cdot long$ at the i_k'th position is bigger than the number at the $i_k + 1$'st position.

For more details on this topic, consult for example [**5**] (pages 18-21).

2. Young tableaux and set partitions

In this section we introduce the combinatorial objects which we will use to construct the representations of the two algebras that we will study in the next chapters.

2.1. Combinatorics of Young tableaux. Let $\mathbb{N}^0$ denote the nonnegative integers. We next recall the combinatorics of Young diagrams and tableaux. A *composition* $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ of $n \in \mathbb{N}^0$ is a finite sequence in $\mathbb{N}^0$ with sum n. The μ_i's are called the parts of μ. A *partition* of n is a composition whose parts are non-increasing. We write $\mu \models n$ and $\lambda \vdash n$ if μ is a composition of n and λ is a partition of n. In these cases we set $|\mu| := n$ and $|\lambda| := n$ and define the length of μ or λ as the number of parts of μ or λ. If $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ is a composition of length l we define the opposite composition μ^{op} as $\mu^{op} := (\mu_l, \ldots, \mu_2, \mu_1)$. We denote by $\mathcal{C}omp_n$ the set of compositions of n and by $\mathcal{P}ar_n$ the set of partitions of n. The *Young diagram* of a composition μ is the subset

$$[\mu] = \{(i,j) \mid 1 \le j \le \mu_i \text{ and } i \ge 1\}$$

of $\mathbb{N}^0 \times \mathbb{N}^0$. The elements of $[\mu]$ are called the *nodes* of μ. We represent $[\mu]$ as an array of boxes in the plane, identifying each node with a box. For example, if $\mu = (3,2,4)$ then

$$[\mu] = \ydiagram{3,2,4}\,.$$

For $\mu \models n$ we define a μ-*tableau* as a bijection $\mathfrak{t} : [\mu] \to \mathbf{n}$. We identify μ-*tableaux* with labellings of the nodes of $[\mu]$: for example, if $\mu = (1,3)$ then $\begin{ytableau} 1 \\ 2 & 3 & 4 \end{ytableau}$ is a μ-tableau. If $\mathfrak{t}$ is a μ-tableau we write $Shape(\mathfrak{t}) := \mu$.

We say that a μ-tableau $\mathfrak{t}$ is *row standard* if the entries in $\mathfrak{t}$ increase from left to right in each row and we say that $\mathfrak{t}$ is *standard* if $\mathfrak{t}$ is row standard and the entries also increase from top to bottom in each column. The set of standard λ-tableaux is denoted $\mathrm{Std}(\lambda)$ and we write $d_\lambda := |\mathrm{Std}(\lambda)|$ for its cardinality. For example, $\begin{ytableau} 2 & 3 & 5 \\ 1 & 4 \end{ytableau}$ is row standard and $\begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \end{ytableau}$ is standard. For a composition of μ of n we denote by $\mathfrak{t}^\mu$ the standard tableau in which the integers $1,2,\dots,n$ are entered in increasing order from left to right along the rows of $[\mu]$. For example, if $\mu = (2,4)$ then $\mathfrak{t}^\mu = \begin{ytableau} 1 & 2 \\ 3 & 4 & 5 & 6 \end{ytableau}$.

The symmetric group $\mathfrak{S}_n$ acts on the right on the set of μ-tableaux by permuting the entries inside a given tableau. Let $\mathfrak{s}$ be a row standard λ-tableau. We denote by $d(\mathfrak{s})$ the unique element of $\mathfrak{S}_n$ such that $\mathfrak{s} = \mathfrak{t}^\lambda d(\mathfrak{s})$. The *Young subgroup* $\mathfrak{S}_\mu$ associated with a composition μ is the row stabilizer of $\mathfrak{t}^\mu$. Let $\mu = (\mu_1,\dots,\mu_k)$ and $\nu = (\nu_1,\dots,\nu_l)$ be compositions. We write $\mu \trianglerighteq \nu$ if for all $i \geq 1$ we have

$$\sum_{j=1}^{i} \mu_j \geq \sum_{j=1}^{i} \nu_j$$

where we add zero parts $\mu_i := 0$ and $\nu_i := 0$ at the end of μ and ν so that the sums are always defined. This is the dominance order on compositions. We extend it to row standard tableaux as follows. Given a row standard tableau $\mathfrak{t}$ of some shape and an integer $m \leq n$, we let $\mathfrak{t} \downarrow m$ be the tableau obtained from $\mathfrak{t}$ by deleting all nodes with entries greater than m. Then, for a pair of μ-tableaux $\mathfrak{s}$ and $\mathfrak{t}$ we write $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if $Shape(\mathfrak{s} \downarrow m) \trianglerighteq Shape(\mathfrak{t} \downarrow m)$ for all $m = 1,\dots,n$. We write $\mathfrak{s} \triangleright \mathfrak{t}$ if $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$. This defines the dominance order on row standard tableaux. It is only a partial order, for example

$$\begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau} \triangleright \begin{ytableau} 2 & 4 \\ 3 & 5 \\ 1 \end{ytableau} \quad \text{and} \quad \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau} \triangleright \begin{ytableau} 4 & 5 \\ 1 & 3 \\ 2 \end{ytableau}$$

whereas $\begin{ytableau} 2 & 4 \\ 3 & 5 \\ 1 \end{ytableau}$ and $\begin{ytableau} 4 & 5 \\ 1 & 3 \\ 2 \end{ytableau}$ are incomparable.

We have that $\mathfrak{t}^\lambda \trianglerighteq \mathfrak{t}$ for all row standard λ-tableau $\mathfrak{t}$.

An r-*multicomposition*, or simply a *multicomposition*, of n is an ordered r-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ of compositions $\lambda^{(k)}$ such that $\sum_{i=1}^r |\lambda^{(i)}| = n$. We call $\lambda^{(k)}$ the k'th component of $\boldsymbol{\lambda}$, note that it may be empty. An r-*multipartition*, or simply a *multipartition*, is a multicomposition whose components are partitions. The nodes of a multicomposition are labelled by tuples (x,y,k) with k giving the number of the component and (x,y) the node of that component. For the multicomposition $\boldsymbol{\lambda}$ the set of nodes is denoted $[\boldsymbol{\lambda}]$. This is the Young diagram for $\boldsymbol{\lambda}$ and is represented graphically as the r-tuple of Young diagrams of the components. For example, the Young diagram of $\boldsymbol{\lambda} = ((2,3),(3,1),(1,1,1))$ is

$$\left(\ydiagram{2,3}\,,\ \ydiagram{3,1}\,,\ \ydiagram{1,1,1}\right).$$

We denote by $Comp_{r,n}$ the set of r-multicompositions of n and by $Par_{r,n}$ the set of r-multipartitions of n. Let $\boldsymbol{\lambda}$ be a multicomposition of n. A $\boldsymbol{\lambda}$-*multitableau* is a bijection $\mathfrak{t} : [\boldsymbol{\lambda}] \to \mathbf{n}$ which may once again be identified with a filling of $[\boldsymbol{\lambda}]$ using the numbers from $\mathbf{n}$. The restriction of $\mathfrak{t}$ to $\lambda^{(i)}$ is called the i'th component of $\mathfrak{t}$ and we write $\mathfrak{t} = (\mathrm{t}^{(1)}, \mathrm{t}^{(2)}, \ldots, \mathrm{t}^{(r)})$ where $\mathrm{t}^{(i)}$ is the i'th component of $\mathfrak{t}$. We say that $\mathfrak{t}$ is *row standard* if all its components are row standard, and *standard* if all its components are standard tableaux. If $\mathfrak{t}$ is a $\boldsymbol{\lambda}$-multitableau we write $Shape(\mathfrak{t}) = \boldsymbol{\lambda}$. The set of all standard $\boldsymbol{\lambda}$-multitableaux is denoted by $\mathrm{Std}(\boldsymbol{\lambda})$. In the examples

$$\mathfrak{t} = \left(\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau}\,,\ \begin{ytableau} 6 \\ 7 \\ 8 & 9 \end{ytableau}\right) \qquad \mathfrak{s} = \left(\begin{ytableau} 2 & 7 & 8 \\ 1 & 4 \end{ytableau}\,,\ \begin{ytableau} 5 & 6 \end{ytableau}\,,\ \begin{ytableau} 3 \\ 9 \end{ytableau}\right) \tag{2.1}$$

$\mathfrak{t}$ is a standard multitableau whereas $\mathfrak{s}$ is only a row standard tableau. We denote by $\mathfrak{t}^{\boldsymbol{\lambda}}$ the $\boldsymbol{\lambda}$-multitableau in which $1, 2, \ldots, n$ appear in order along the rows of the first component, then along the rows of the second component, and so on. For example, in (2.1) we have that $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} = ((3,2),(1,1,2))$. For each multicomposition $\boldsymbol{\lambda}$ we define the Young subgroup $\mathfrak{S}_{\boldsymbol{\lambda}}$ as the row stabilizer of $\mathfrak{t}^{\boldsymbol{\lambda}}$.

Let $\mathfrak{s}$ be a row standard $\boldsymbol{\lambda}$-multitableau. We denote by $d(\mathfrak{s})$ the unique element of $\mathfrak{S}_n$ such that $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}} d(\mathfrak{s})$. The set formed by these elements is a complete set of right coset representatives of $\mathfrak{S}_{\boldsymbol{\lambda}}$ in $\mathfrak{S}_n$. Moreover

$$\{d(\mathfrak{s}) \mid \mathfrak{s} \text{ is a row standard } \boldsymbol{\lambda}\text{-multitableau}\}$$

is a *distinguished set of right coset representatives*, that is $\ell(wd(\mathfrak{s})) = \ell(w) + \ell(d(\mathfrak{s}))$ for $w \in \mathfrak{S}_{\boldsymbol{\lambda}}$.

Let $\boldsymbol{\lambda}$ be a multicomposition of n and let $\mathfrak{t}$ be a $\boldsymbol{\lambda}$-multitableau. For $j = 1, \ldots, n$ we write $p_{\mathfrak{t}}(j) := k$ if j appears in the k'th component $\mathrm{t}^{(k)}$ of $\mathfrak{t}$. We call $p_{\mathfrak{t}}(j)$ the *position* of j in $\mathfrak{t}$. When $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$, we write $p_{\boldsymbol{\lambda}}(j)$ for $p_{\mathfrak{t}^{\boldsymbol{\lambda}}}(j)$ and say that a $\boldsymbol{\lambda}$-multitableau $\mathfrak{t}$ is of the *initial kind* if $p_{\mathfrak{t}}(j) = p_{\boldsymbol{\lambda}}(j)$ for all $j = 1, \ldots, n$.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)})$ be multicompositions of n. We write $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if $\lambda^{(i)} \trianglerighteq \mu^{(i)}$ for all $i = 1, \ldots, n$, this is our dominance order on $Comp_{r,n}$. If $\mathfrak{s}$ and $\mathfrak{t}$ are row standard multitableaux and $m = 1, \ldots, n$ we define $\mathfrak{s} \downarrow m$ and $\mathfrak{t} \downarrow m$ as for usual tableaux and write $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if $Shape(\mathfrak{s} \downarrow m) \trianglerighteq Shape(\mathfrak{t} \downarrow m)$ for all m.

It should be noted that our dominance order $\trianglerighteq$ is different from the dominance order on multicompositions and multitableaux that is used in some parts of the literature, for example in [**11**]. Let us denote by $\succeq$ the order used in [**11**]. Then we have that

$$\left(\begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}\,,\ \begin{ytableau} 5 \end{ytableau}\right) \succeq \left(\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\,,\ \begin{ytableau} 4 & 5 \end{ytableau}\right)$$

whereas these multitableaux are incomparable with respect to $\trianglerighteq$. On the other hand, if $\mathfrak{s}$ and $\mathbf{t}$ are multitableaux of the same shape and $p_{\mathfrak{s}}(j) = p_{\mathbf{t}}(j)$ for all j, then we have that $\mathfrak{s} \trianglerighteq \mathbf{t}$ if and only if $\mathfrak{s} \succeq \mathbf{t}$.

To each r-multicomposition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ we associate a composition $\|\boldsymbol{\lambda}\|$ of length r as follows

$$\left\|\boldsymbol{\lambda}\right\| := (|\lambda^{(1)}|, \ldots, |\lambda^{(r)}|). \tag{2.2}$$

Let $\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ be the associated Young subgroup. Then $w \in \mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ iff $\mathbf{t}^{\boldsymbol{\lambda}} w$ is of the initial kind. For any $\boldsymbol{\lambda}$-multitableau $\mathfrak{s}$ there is a decomposition of $d(\mathfrak{s})$ with respect to $\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$, that is

$$d(\mathfrak{s}) = d(\mathfrak{s}_0)\, w_{\mathfrak{s}}, \;\text{ where }\; d(\mathfrak{s}_0) \in \mathfrak{S}_{\|\boldsymbol{\lambda}\|} \;\text{ and }\; l(d(\mathfrak{s})) = l(d(\mathfrak{s}_0)) + l(w_{\mathfrak{s}}). \tag{2.3}$$

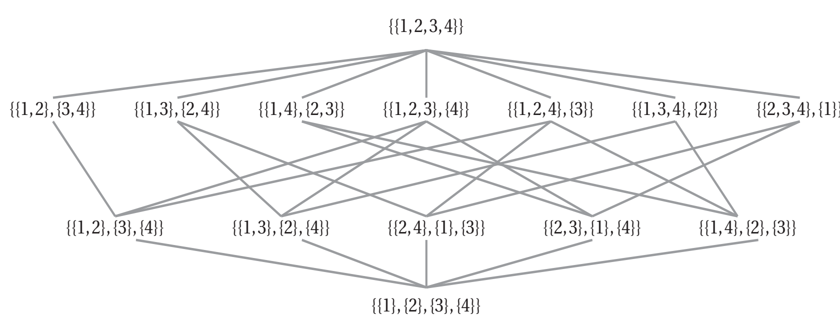
We define in this situation $\mathfrak{s}_0 = \mathbf{t}^{\boldsymbol{\lambda}} d(\mathfrak{s}_0)$; it is of the initial kind. Let $\mathbf{t}$ be another multitableau of shape $\boldsymbol{\mu}$ and let $d(\mathbf{t}) = d(\mathbf{t}_0)\, w_{\mathbf{t}}$ be its decomposition. Suppose that $\left\|\boldsymbol{\lambda}\right\| = \left\|\boldsymbol{\mu}\right\|$ and that $w_{\mathfrak{s}} = w_{\mathbf{t}}$. Then we have the following compatibility property with respect to the dominance order

$$\mathfrak{s} \trianglerighteq \mathbf{t} \text{ if and only if } \mathfrak{s}_0 \trianglerighteq \mathbf{t}_0. \tag{2.4}$$

Let $\alpha := \left\|\boldsymbol{\lambda}\right\|$. Let $y \in \mathfrak{S}_n$ be as short as possible such that $\mathfrak{s} y$ is of the initial kind and set $\mathrm{t} := \mathrm{t}^{\alpha} d(\mathfrak{s}) y$. Then $d(\mathfrak{s}_0) = d(\mathrm{t})$ and $w_{\mathfrak{s}} = y^{-1}$. If $y = s_{i_1} s_{i_2} \ldots s_{i_k}$ is reduced expression for y then for all j we have that i_j and $i_j + 1$ occur in distinct components of $\mathfrak{s} s_{i_1} s_{i_2} \ldots s_{i_{j-1}}$ (with $i_j + 1$ to the left of i_j) as can be seen using the inversion description of the length function on $\mathfrak{S}_n$, and a similar property holds for $w_{\mathfrak{s}}$.

2.2. Set partitions. Recall that a set of subsets $A = \{I_1, I_2, \ldots I_k\}$ of $\mathbf{n}$ is called a *set partition* of $\mathbf{n}$ if the I_j's are nonempty, disjoint and have union $\mathbf{n}$. For example, $\{\{1,2,3\},\{4\}\}$, $\{\{1,3\},\{2\},\{4\}\}$ and $\{\{1,3\},\{2,4\}\}$ are set partitions of $\mathbf{4}$. We refer to the I_i's as the blocks of A and we denote by $\mathcal{SP}_n$ the set of all set partitions of $\mathbf{n}$. There is a natural poset structure on $\mathcal{SP}_n$ defined as follows. Suppose that $A = \{I_1, I_2, \ldots, I_k\} \in \mathcal{SP}_n$ and $B = \{J_1, I_2, \ldots, J_l\} \in \mathcal{SP}_n$. Then the order relation on $\mathcal{SP}_n$ is given by $A \subseteq B$ if each J_j is a union of some of the I_i's.

The following Hasse diagram illustrates the order relation $\subseteq$ in $\mathcal{SP}_4$



Finally, to each multicomposition we can associate an unique set partition as we will explain next. For a composition $\mu = (\mu_1, \ldots, \mu_k)$ we define the *reduced composition* $\operatorname{red}\mu$ as the composition obtained from μ by deleting all zero parts $\mu_i = 0$ from μ. We say that a composition μ is reduced if $\mu = \mathrm{red}\,\mu$.

For any reduced composition $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ we introduce the set partition $A_\mu := (I_1, I_2, \dots, I_k)$ by filling in the numbers consecutively, that is

$$I_1 := \{1, 2, \dots, \mu_1\},\ I_2 := \{\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2\},\ etc. \tag{2.5}$$

and for a multicomposition $\boldsymbol{\lambda} \in Comp_{r,n}$ we define $A_{\boldsymbol{\lambda}} := A_{\mathrm{red}\|\boldsymbol{\lambda}\|} \in \mathcal{SP}_n$. For example, for $\boldsymbol{\lambda} = ((0), (1, 2), (0), (1, 1), (2, 1))$ we have that $A_{\boldsymbol{\lambda}} = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$.

3. Cellular algebras

The following definition appeared for the first time in [**16**].

DEFINITION 1.1. *Let $\mathcal{R}$ be an integral domain. Suppose that A is an $\mathcal{R}$-algebra which is free as an $\mathcal{R}$-module. Suppose that $(\Lambda, \geq)$ is a poset and that for each $\lambda \in \Lambda$ there is a finite indexing set $T(\lambda)$ (the 'λ-tableaux') and elements $c^\lambda_{\mathfrak{s}\mathfrak{t}} \in A$ such that*

$$\mathcal{C} = \{c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$$

is an $\mathcal{R}$-basis of A. The pair $(\mathcal{C}, \Lambda)$ is a cellular basis of A if

- **(i)** *The $\mathcal{R}$-linear map $* : A \to A$ determined by $(c^\lambda_{\mathfrak{s}\mathfrak{t}})^* = c^\lambda_{\mathfrak{t}\mathfrak{s}}$ for all $\lambda \in \lambda$ and all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ is an algebra anti-automorphism of A.*
- **(ii)** *For any $\lambda \in \Lambda$, $\mathfrak{t} \in T(\lambda)$ and $a \in A$ there exist $r_\mathfrak{v} \in \mathcal{R}$ such that for all $\mathfrak{s} \in T(\lambda)$*

$$c^\lambda_{\mathfrak{s}\mathfrak{t}} a \equiv \sum_{\mathfrak{v} \in T(\lambda)} r_\mathfrak{v} c^\lambda_{\mathfrak{s}\mathfrak{v}} \mod A^\lambda$$

where A^λ is the $\mathcal{R}$-submodule of A with basis $\{c^\mu_{\mathfrak{u}\mathfrak{v}} \mid \mu \in \Lambda, \mu > \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in T(\mu)\}$.

If A has a cellular basis we say that A is a cellular algebra and $(\Lambda, T, \mathcal{C}, *)$ is called the "cell datum" of A.

The following is one of the motivational examples of this definition.

EXAMPLE 1. *Let $\mathcal{H}_n(q)$ the associative S-algebra given by generators $h_1, h_2, \dots, h_{n-1}$ subject to the relations:*

$$h_i h_j = h_j h_i \qquad \text{for } |i - j| > 1 \tag{3.1}$$

$$h_i h_{i+1} h_i = h_{i+1} h_i h_{i+1} \qquad \text{for } i = 1, 2, \dots, n-2 \tag{3.2}$$

$$h_i^2 = 1 + (q - q^{-1}) h_i \qquad \text{for } i = 1, 2, \dots, n-1. \tag{3.3}$$

The algebra $\mathcal{H}_n(q)$ is called the Iwahori-Hecke algebra of type A_{n-1}.

Graham and Lehrer showed that $\mathcal{H}_n(q)$ is a cellular algebra using the Kazhdan-Lusztig basis for $\mathcal{H}_n(q)$ and the Robinson-Schensted algorithm. However, in this thesis we are more interested in another cellular basis for $\mathcal{H}_n(q)$, which was introduced by Murphy in [**37**]*, independently of* [**16**].

Consider $\Lambda = \mathcal{P}ar_n$ with the dominance order, $T(\lambda) = \operatorname{Std}(\lambda)$ (for each $\lambda \in \Lambda$) and the anti-automorphism $* : h_w \to h_{w^{-1}}$. Let $w \in \mathfrak{S}_n$ and suppose that $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for w. Then, we define

$$h_w := h_{i_1} h_{i_2} \cdots h_{i_k} \quad \text{and} \quad h_{\mathrm{id}} = 1 \in S$$

From Matsumoto's theorem the elements $h_w \in \mathcal{H}_n(q)$ are well-defined, that is h_w is independent of the choice of reduced expression for w. The Murphy cellular basis for $\mathcal{H}_n(q)$ is given by the set

$$\mathcal{C}_{\mathcal{H}_n} = \{x^{\lambda}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda),\ \lambda \in \mathcal{P}ar_n\} \tag{3.4}$$

where $x^{\lambda}_{\mathfrak{s}\mathfrak{t}} := h^*_{d(\mathfrak{s})} x_\lambda h_{d(\mathfrak{t})}$ and $x_\lambda := \sum_{w \in \mathfrak{S}_\lambda} q^{\ell(w)} h_w$.

For example, we describe the Murphy basis for $\mathcal{H}_4(q)$. In this case, $\Lambda = \{(1^4), (2,1^2), (2^2), (3,1), (4)\}$ and

$$T\left((1^4)\right) = \left\{ \mathfrak{s}_1 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array} \right\}, \quad T\left((2,1^2)\right) = \left\{ \mathfrak{s}_2 = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}, \mathfrak{s}_3 = \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}, \mathfrak{s}_4 = \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} \right\}, \quad T\left((2^2)\right) = \left\{ \mathfrak{s}_5 = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}, \mathfrak{s}_6 = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \right\}$$

$$T\left((3,1)\right) = \left\{ \mathfrak{s}_7 = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}, \mathfrak{s}_8 = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}, \mathfrak{s}_9 = \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array} \right\}, \quad T\left((1^4)\right) = \left\{ \mathfrak{s}_{10} = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array} \right\}$$

Considering these assignations, the elements of the cellular basis are:

- $x^{(1^4)}_{\mathfrak{s}_1\mathfrak{s}_1} = 1$

- $x^{(2,1^3)}_{\mathfrak{s}_2\mathfrak{s}_2} = 1 + qh_1$
- $x^{(2,1^3)}_{\mathfrak{s}_2\mathfrak{s}_3} = (1 + qh_1)h_2$
- $x^{(2,1^3)}_{\mathfrak{s}_2\mathfrak{s}_4} = (1 + qh_1)h_2 h_3$
- $x^{(2,1^3)}_{\mathfrak{s}_3\mathfrak{s}_2} = h_2(1 + qh_1)$
- $x^{(2,1^3)}_{\mathfrak{s}_3\mathfrak{s}_2} = h_2(1 + qh_1)h_2$
- $x^{(2,1^3)}_{\mathfrak{s}_3\mathfrak{s}_2} = h_2(1 + qh_1)h_2 h_3$
- $x^{(2,1^3)}_{\mathfrak{s}_4\mathfrak{s}_2} = h_3 h_2(1 + qh_1)$
- $x^{(2,1^3)}_{\mathfrak{s}_4\mathfrak{s}_3} = h_3 h_2(1 + qh_1)h_2$
- $x^{(2,1^3)}_{\mathfrak{s}_4\mathfrak{s}_4} = h_3 h_2(1 + qh_1)h_2 h_3$

- $x^{(2^2)}_{\mathfrak{s}_5\mathfrak{s}_5} = (1 + qh_1)(1 + qh_3)$
- $x^{(2^2)}_{\mathfrak{s}_5\mathfrak{s}_6} = (1 + qh_1)(1 + qh_3)h_2$
- $x^{(2^2)}_{\mathfrak{s}_6\mathfrak{s}_5} = h_2(1 + qh_1)(1 + qh_3)$
- $x^{(2^2)}_{\mathfrak{s}_6\mathfrak{s}_6} = h_2(1 + qh_1)(1 + qh_3)h_2$

- $x^{(3,1)}_{\mathfrak{s}_7\mathfrak{s}_7} = \sum_{w \in \mathfrak{S}_3} q^{\ell(w)} h_w$
- $x^{(3,1)}_{\mathfrak{s}_7\mathfrak{s}_8} = \left(\sum_{w \in \mathfrak{S}_3} q^{\ell(w)} h_w \right) h_2$
- $x^{(3,1)}_{\mathfrak{s}_7\mathfrak{s}_9} = \left(\sum_{w \in \mathfrak{S}_3} q^{\ell(w)} h_w \right) h_2 h_3$
- $x^{(3,1)}_{\mathfrak{s}_8\mathfrak{s}_7} = h_2 \left(\sum_{w \in \mathfrak{S}_3} q^{\ell(w)} h_w \right)$
- $x^{(3,1)}_{\mathfrak{s}_8\mathfrak{s}_8} = h_2 \left(\sum_{w \in \mathfrak{S}_3} q^{\ell(w)} h_w \right) h_2$
- $x^{(3,1)}_{\mathfrak{s}_8\mathfrak{s}_9} = h_2 \left(\sum_{w \in \mathfrak{S}_3} q^{\ell(w)} h_w \right) h_2 h_3$
- $x^{(3,1)}_{\mathfrak{s}_9\mathfrak{s}_7} = h_3 h_2 \left(\sum_{w \in \mathfrak{S}_3} q^{\ell(w)} h_w \right)$
- $x^{(3,1)}_{\mathfrak{s}_9\mathfrak{s}_8} = h_3 h_2 \left(\sum_{w \in \mathfrak{S}_3} q^{\ell(w)} h_w \right) h_2$
- $x^{(3,1)}_{\mathfrak{s}_9\mathfrak{s}_9} = h_3 h_2 \left(\sum_{w \in \mathfrak{S}_3} q^{\ell(w)} h_w \right) h_2 h_3$

- $x^{(4)}_{\mathfrak{s}_{10}\mathfrak{s}_{10}} = \sum_{w \in \mathfrak{S}_4} q^{\ell(w)} h_w$

EXAMPLE 2. *Let $\mathcal{R}$ an integral domain. Then, the matrix algebra* $\mathrm{Mat}_n(R)$ *is a cellular algebra with cellular basis*

$$\left\{M^n_{ij}\middle| i,j\in\{1,2,\ldots,n\}\right\}$$

Here $\Lambda=\{n\}$, $T(n)=\{1,2,\ldots,n\}$ *and* $M^*:=M^t$ *is the transpose of matrix* M. *Furthermore, it is easy to verify that the axiom (**ii**) of* Definition *1.1 holds by the fact that* $M_{ik}M_{lj}=\delta_{kl}M_{ij}$, *where* δ_{kl} *is the Dirac's delta function.*

In the sequel, we consider A as a cellular algebra with cellular datum $(\Lambda,T,\mathcal{C},*)$ as in Definition 1.1.

DEFINITION 1.2. *For each* $\lambda\in\Lambda$ *we define the cell module* $C(\lambda)$ *as the left* A*-module which is free as* $\mathcal{R}$*-module, with basis* $\{c^\lambda_{\mathfrak{s}}\mid s\in T(\lambda)\}$ *and* A*-left action given by*

$$ac^\lambda_{\mathfrak{s}}=\sum_{\mathfrak{t}\in T(\lambda)} r_{\mathfrak{t}}c^\lambda_{\mathfrak{t}}$$

where the scalars $r_{\mathfrak{t}}\in\mathcal{R}$ *are the elements appearing in* Definition 1.1(**ii**).

From the defining axioms it is not hard to prove that for each $\lambda\in\Lambda$ there exists a symmetric and associative bilinear form

$$\langle\,,\,\rangle_\lambda : C(\lambda)\times C(\lambda)\to\mathcal{R} \tag{3.5}$$

such that $\langle c^\lambda_{\mathfrak{s}},c^\lambda_{\mathfrak{t}}\rangle_\lambda$ for all $\mathfrak{s},\mathfrak{t}\in T(\lambda)$ is determinate by

$$\langle c^\lambda_{\mathfrak{s}},c^\lambda_{\mathfrak{t}}\rangle_\lambda c^\lambda_{\mathfrak{u}\mathfrak{v}}\equiv c^\lambda_{\mathfrak{u}\mathfrak{s}}c^\lambda_{\mathfrak{t}\mathfrak{v}}\mod A^\lambda$$

where $\mathfrak{u}$ and $\mathfrak{v}$ are any elements of $\mathfrak{t}(\lambda)$.

Since $\langle\,,\,\rangle_\lambda$ is associative we have that the set

$$\mathrm{rad}(\lambda):=\{x\in C(\lambda)\mid\langle x,y\rangle_\lambda=0,\ \text{ for all } y\in C(\lambda)\} \tag{3.6}$$

is an A-submodule of $C(\lambda)$ (see [**35**, Proposition 2.9]). Defining $D(\lambda):=C(\lambda)/\mathrm{rad}(\lambda)$ and $\Lambda_0:=\{\lambda\in\Lambda\mid D(\lambda)\neq 0\}$ we have the following result

THEOREM 1.1 (Graham-Lehrer). *If* $\mathcal{R}$ *is a field and* Λ *is finite, then* $\{D(\lambda)\mid\lambda\in\Lambda_0\}$ *is a complete set of pairwise non-isomorphic simple* A*-modules.*

The above theorem classifies all the simple A-modules for a finite dimensional cellular algebra A, but in practice, it is not easy to determinate the set Λ_0. For example, in [**37**] Murphy characterizes this set for the Hecke algebra of type A_{n-1} in terms of the so-called e-partitions. In particular, if $\mathcal{F}(S)$ is the field of fractions of S we have that $D(\lambda)=C(\lambda)$ for all $\lambda\in\mathcal{P}ar_n$.

CHAPTER 2

Representation theory of the Yokonuma-Hecke algebra

1. Yokonuma Hecke algebra

DEFINITION 2.1. *Let n be a positive integer. The Yokonuma-Hecke algebra, denoted $\mathcal{Y}_{r,n} = \mathcal{Y}_{r,n}(q)$, is the associative R-algebra generated by the elements $g_1, \dots, g_{n-1}, t_1, \dots, t_n$, subject to the following relations:*

$$t_i^r = 1 \qquad \text{for all } i \tag{1.1}$$

$$t_i t_j = t_j t_i \qquad \text{for all } i, j \tag{1.2}$$

$$t_j g_i = g_i t_{j s_i} \qquad \text{for all } i, j \tag{1.3}$$

$$g_i g_j = g_j g_i \qquad \text{for } |i-j| > 1 \tag{1.4}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad \text{for all } i = 1, \dots, n-2 \tag{1.5}$$

together with the quadratic relation

$$g_i^2 = 1 + (q - q^{-1}) e_i g_i \qquad \text{for all } i \tag{1.6}$$

where

$$e_i := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s}. \tag{1.7}$$

Note that since r is invertible in R, the element $e_i \in \mathcal{Y}_{r,n}(q)$ makes sense.

One checks that e_i is an idempotent and that g_i is invertible in $\mathcal{Y}_{r,n}(q)$ with inverse $g_i^{-1} = g_i + (q^{-1} - q)e_i$, as it is shown in the following calculation

$$e_i^2 = \frac{1}{r^2} \sum_{s=0}^{r-1} \sum_{p=0}^{r-1} t_i^{s+p} t_{i+1}^{-(s+p)} = \frac{1}{r^2} \sum_{s=0}^{r-1} \sum_{l=s}^{r-1+s} t_i^l t_{i+1}^{-l} = \frac{1}{r^2} \cdot r \sum_{l=0}^{r-1} t_i^l t_{i+1}^{-l} = \frac{1}{r} \sum_{l=0}^{r-1} t_i^l t_{i+1}^{-l} = e_i$$

and

$$g_i(g_i + (q^{-1} - q)e_i) = 1 + (q - q^{-1}) g_i e_i + (q^{-1} - q) g_i e_i = 1.$$

The study of the representation theory $\mathcal{Y}_{r,n}(q)$ is one of the main themes of the present thesis. $\mathcal{Y}_{r,n}(q)$ can be considered as a generalization of the usual Iwahori-Hecke algebra $\mathcal{H}_n = \mathcal{H}_n(q)$ of type A_{n-1} since $\mathcal{Y}_{1,n}(q) = \mathcal{H}_n(q)$. In general $\mathcal{H}_n(q)$ is a canonical quotient of $\mathcal{Y}_{r,n}(q)$ via the ideal generated by all the $t_i - 1$'s. On the other hand, as a consequence of the results of the present thesis, $\mathcal{H}_n(q)$ also appears as a subalgebra of $\mathcal{Y}_{r,n}(q)$ although not canonically.

$\mathcal{Y}_{r,n}(q)$ was introduced by Yokonuma in the sixties as the endomorphism algebra of a module for the Chevalley group of type A_{n-1}, generalizing the usual Iwahori-Hecke algebra construction, see [**44**]. This also gave rise to a presentation for $\mathcal{Y}_{r,n}(q)$. A different presentation for $\mathcal{Y}_{r,n}(q)$, widely used in the literature, was found by Juyumaya. The presentation given above appeared first in [**8**] and differs slightly from Juyumaya's presentation. In Juyumaya's presentation another variable u is used and the quadratic relation (1.6) takes the form $\tilde{g}_i^2 = 1+(u-1)e_i(\tilde{g}_i+1)$. The relationship between the two presentations is given by $u = q^2$ and

$$\tilde{g}_i = g_i + (q-1)e_i g_i, \tag{1.8}$$

or equivalently $g_i = \tilde{g}_i + (q^{-1}-1)e_i\tilde{g}_i$, see eg. [**9**].

In this thesis we shall be interested in the general, not necessarily semisimple, representation theory of $\mathcal{Y}_{r,n}(q)$ and shall therefore need base change of the ground ring. Let $\mathcal{K}$ be a commutative ring, with elements $q, \xi \in \mathcal{K}^\times$. Suppose moreover that ξ is an r'th root of unity and that r and $\prod_{0\le i<j\le r-1}(\xi^i - \xi^j)$ are invertible in $\mathcal{K}$ (for example $\mathcal{K}$ a field with $r, \xi \in \mathcal{K}^\times$ and ξ a primitive r'th root of unity). Then we can make $\mathcal{K}$ into an R-algebra by mapping $q \in R$ to $q \in \mathcal{K}$, and $\xi \in R$ to $\xi \in \mathcal{K}$. This gives rise to the specialized Yokonuma-Hecke algebra

$$\mathcal{Y}_{r,n}^{\mathcal{K}}(q) = \mathcal{Y}_{r,n}(q) \otimes_R \mathcal{K}.$$

Let $w \in \mathfrak{S}_n$ and suppose that $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ is a reduced expression for w. Then by the relations the element $g_w := g_{i_1}g_{i_2}\cdots g_{i_m}$ does not depend on the choice of the reduced expression for w. We use the convention that $g_1 := 1$. In [**26**] Juyumaya proved that the following set is an R-basis for $\mathcal{Y}_{r,n}(q)$

$$\mathcal{B}_{r,n} = \{t_1^{k_1}t_2^{k_2}\cdots t_n^{k_n} g_w \mid w \in \mathfrak{S}_n,\ k_1,\dots,k_n \in \mathbb{Z}/r\mathbb{Z}\}. \tag{1.9}$$

In particular, $\mathcal{Y}_{r,n}(q)$ is a free R-module of rank $r^n n!$. Similarly, $\mathcal{Y}_{r,n}^{\mathcal{K}}(q)$ is a free over $\mathcal{K}$ of rank $r^n n!$.

Let us introduce some useful elements of $\mathcal{Y}_{r,n}(q)$ (or $\mathcal{Y}_{r,n}^{\mathcal{K}}(q)$). For $1 \le i,j \le n$ we define e_{ij} by

$$e_{ij} := \frac{1}{r}\sum_{s=0}^{r-1} t_i^s t_j^{-s}. \tag{1.10}$$

These e_{ij}'s are idempotents and $e_{ii} = 1$ and $e_{i,i+1} = e_i$. Moreover $e_{ij} = e_{ji}$ and it is easy to verify from (1.3) that

$$e_{ij} = g_i g_{i+1}\cdots g_{j-2}e_{j-1}g_{j-2}^{-1}\cdots g_{i+1}^{-1}g_i^{-1} \qquad \text{for } i<j. \tag{1.11}$$

From (1.1)-(1.3) one obtains that

$$t_i e_{ij} = t_j e_{ij} \qquad \text{for all } i,j \tag{1.12}$$

$$e_{ij}e_{kl} = e_{kl}e_{ij} \qquad \text{for all } i,j,k,l \tag{1.13}$$

$$e_{ij}g_k = g_k e_{is_k, js_k} \text{ for all } i,j \text{ and } k = 1,\dots,n-1. \tag{1.14}$$

For any nonempty subset $I \subset \mathbf{n}$ we extend the definition of e_{ij} to E_I by setting

$$E_I := \prod_{i,j \in I, i<j} e_{ij} \tag{1.15}$$

where we use the convention that $E_I := 1$ if $|I| = 1$.

We need a further generalization of this. For any set partition $A = \{I_1, I_2, \dots, I_k\} \in \mathcal{SP}_n$ we define

$$E_A := \prod_j E_{I_j}. \tag{1.16}$$

Extending the right action of $\mathfrak{S}_n$ on $\mathbf{n}$ to a right action on $\mathcal{SP}_n$ via $Aw := \{I_1 w, \dots, I_k w\} \in \mathcal{SP}_n$ for $w \in \mathfrak{S}_n$, we have the following Lemma.

LEMMA 1. *For $A \in \mathcal{SP}_n$ and $w \in \mathfrak{S}_n$ as above, we have that*

$$E_A g_w = g_w E_{Aw}.$$

In particular, if w leaves invariant every block of A, or more generally permutes certain of the blocks of A (of the same size), then E_A and g_w commute.

PROOF. This is immediate from (1.14) and the definitions. □

2. Tensorial representation of $\mathcal{Y}_{r,n}(q)$

In this section we obtain our first results by constructing a tensor space module for the Yokonuma-Hecke algebra which we show is faithful. From this we deduce that the Yokonuma-Hecke algebra is in fact isomorphic to a specialization of the 'modified Ariki-Koike' algebra, that was introduced by Shoji in [**42**] and studied for example in [**41**].

DEFINITION 2.2. *Let V be the free R-module with basis $\{v_i^t \mid 1 \le i \le n,\ 0 \le t \le r-1\}$. Then we define operators $\mathbf{T} \in \mathrm{End}_R(V)$ and $\mathbf{G} \in \mathrm{End}_R(V^{\otimes 2})$ as follows:*

$$(v_i^t)\mathbf{T} := \xi^t v_i^t \tag{2.1}$$

and

$$(v_i^t \otimes v_j^s)\mathbf{G} := \begin{cases} v_j^s \otimes v_i^t & \text{if } t \neq s \\ q v_i^t \otimes v_j^s & \text{if } t = s,\ i = j \\ v_j^s \otimes v_i^t & \text{if } t = s,\ i > j \\ (q - q^{-1}) v_i^t \otimes v_j^s + v_j^s \otimes v_i^t & \text{if } t = s,\ i < j. \end{cases} \tag{2.2}$$

We extend them to operators $\mathbf{T}_i$ and $\mathbf{G}_i$ acting in the tensor space $V^{\otimes n}$ by letting $\mathbf{T}$ act in the i'th factor and $\mathbf{G}$ in the i'th and $i+1$'st factors, respectively.

Our goal is to prove that these operators define a faithful representation of the Yokonuma-Hecke algebra. We first prove an auxiliary Lemma.

LEMMA 2. *Let* $\mathbf{E}_i$ *be defined by* $\mathbf{E}_i := \dfrac{1}{r}\sum_{m=0}^{r-1}\mathbf{T}_i^m\mathbf{T}_{i+1}^{-m}$. *Consider the map*

$$(v_i^t \otimes v_j^s)\mathbf{E} := \begin{cases} 0 & \text{if } t \neq s \\ v_i^t \otimes v_j^s & \text{if } t = s. \end{cases}$$

Then $\mathbf{E}_i$ *acts in* $V^{\otimes n}$ *as* $\mathbf{E}$ *in the factors* $(i, i+1)$ *and as the identity in the rest.*

PROOF. We have that

$$(v_j^t \otimes v_k^t)\mathbf{T}_i\mathbf{T}_{i+1}^{-1} = \xi^t\xi^{-t}v_j^t \otimes v_k^t = v_j^t \otimes v_k^t.$$

Thus we get immediately that $(v_i^t \otimes v_j^s)\mathbf{E}_i = v_i^t \otimes v_j^s$ if $s = t$. Now, if $s \neq t$ we have that

$$(v_j^t \otimes v_k^s)\mathbf{T}_i\mathbf{T}_{i+1}^{-1} = \xi^t\xi^{-s}v_j^t \otimes v_k^t = \xi^{t-s}v_j^t \otimes v_k^t.$$

Since $0 \leq t, s \leq r-1$, we have that $\xi^{t-s} \neq 1$ which implies that

$$\sum_{m=0}^{r-1}\xi^{m(t-s)} = (\xi^{r(t-s)} - 1)/(\xi^{(t-s)} - 1) = 0$$

and so it follows that $(v_i^t \otimes v_j^s)\mathbf{E} = 0$ if $s \neq t$. □

REMARK 1. *The operators* $\mathbf{G}_i$ *and* $\mathbf{E}_i$ *should be compared with the operators introduced in* [**39**] *in order to obtain a representation of* $\mathcal{E}_n(q)$ *in* $V^{\otimes n}$. *Let us denote by* $\widetilde{\mathbf{G}}_i$ *and* $\widetilde{\mathbf{E}}_i$ *the operators defined in* [**39**]. *Then we have that* $\mathbf{E}_i = \widetilde{\mathbf{E}}_i$ *and*

$$\mathbf{G}_i = \widetilde{\mathbf{G}}_i + (q^{-1} - 1)\widetilde{\mathbf{E}}_i\widetilde{\mathbf{G}}_i$$

corresponding to the change of presentation given in (1.10).

THEOREM 2.1. *There is a representation* ρ *of* $\mathcal{Y}_{r,n}(q)$ *in* $V^{\otimes n}$ *given by* $t_i \to \mathbf{T}_i$ *and* $g_i \to \mathbf{G}_i$.

PROOF. We must check that the operators $\mathbf{T}_i$ and $\mathbf{G}_i$ satisfy the relations (1.1), ..., (1.6) of the Yokonuma-Hecke algebra. Here the relations (1.1) and (1.2) are trivially satisfied since the $\mathbf{T}_i$'s commute. The relation (1.4) is also easy to verify since the operators $\mathbf{G}_i$ and $\mathbf{G}_j$ act as $\mathbf{G}$ in two different consecutive factors if $|i-j| > 1$.

In order to prove the braid relations (1.5) we rely on the fact, obtained in [**39**] Theorem 1, that the operators $\widetilde{\mathbf{G}}_i$'s and $\widetilde{\mathbf{E}}_i$'s satisfy the relations for $\mathcal{E}_n(q)$ (with modified quadratic relation as indicated just below Definition 3.1). In particular, the braid relations $\widetilde{\mathbf{G}}_i\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{G}}_i = \widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{G}}_i\widetilde{\mathbf{G}}_{i+1}$ hold and also $\widetilde{\mathbf{E}}_i\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{G}}_i\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{E}}_i = \widetilde{\mathbf{E}}_{i+1}\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{G}}_i\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{E}}_{i+1}$ holds, as one sees from Definition 3.1. Via Remark 1 we now get that

$$\begin{aligned}
\mathbf{G}_i\mathbf{G}_{i+1}\mathbf{G}_i &= (1 + (q^{-1} - 1)\widetilde{\mathbf{E}}_i)(\widetilde{\mathbf{G}}_i\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{G}}_i + (q^{-1} - 1)\widetilde{\mathbf{G}}_i\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{E}}_{i+1}\widetilde{\mathbf{G}}_i)(1 + (q^{-1} - 1)\widetilde{\mathbf{E}}_i) \\
&= (1 + (q^{-1} - 1)\widetilde{\mathbf{E}}_i)(\widetilde{\mathbf{G}}_i\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{G}}_i + (q^{-1} - 1)\widetilde{\mathbf{G}}_i\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{E}}_{i+1}\widetilde{\mathbf{E}}_i\widetilde{\mathbf{G}}_i)(1 + (q^{-1} - 1)\widetilde{\mathbf{E}}_i) \\
&= (1 + (q^{-1} - 1)\widetilde{\mathbf{E}}_{i+1})(\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{G}}_i\widetilde{\mathbf{G}}_{i+1} + (q^{-1} - 1)\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{G}}_i\widetilde{\mathbf{G}}_{i+1}\widetilde{\mathbf{E}}_i\widetilde{\mathbf{E}}_{i+1})(1 + (q^{-1} - 1)\widetilde{\mathbf{E}}_{i+1}) \\
&= \mathbf{G}_{i+1}\mathbf{G}_i\mathbf{G}_{i+1}
\end{aligned}$$

and (1.5) follows as claimed. In a similar way we get that the $\mathbf{G}_i$'s satisfy the quadratic relation (1.6).

We are then only left with the relation (1.3). We have here three cases to consider:

$$\mathbf{T}_i\mathbf{G}_j = \mathbf{G}_j\mathbf{T}_i \qquad |i-j|>1 \tag{2.3}$$

$$\mathbf{T}_i\mathbf{G}_i = \mathbf{G}_i\mathbf{T}_{i+1} \tag{2.4}$$

$$\mathbf{T}_{i+1}\mathbf{G}_i = \mathbf{G}_i\mathbf{T}_i. \tag{2.5}$$

The case (2.3) clearly holds since the operators $\mathbf{T}_i$ and $\mathbf{G}_j$ act in different factors of the tensor product $v_{i_1}^{j_1}\otimes v_{i_2}^{j_2}\otimes\ldots\otimes v_{i_n}^{j_n}$. In order to verify the other two cases we may assume that $i=1$ and $n=2$. It is enough to evaluate on vectors of the form $v_{i_1}^{j_1}\otimes v_{i_2}^{j_2}\in V^{\otimes 2}$. For $j_1=j_2$ the actions of $\mathbf{T}_1$ and $\mathbf{T}_2$ are given as the multiplication with the same scalar and so the relations (2.4) and (2.5) also hold.

Suppose then finally that $j_1\neq j_2$. We then have that

$$(v_{i_1}^{j_1}\otimes v_{i_2}^{j_2})\mathbf{T}_1\mathbf{G}_1 = \xi^{j_1} v_{i_2}^{j_2}\otimes v_{i_1}^{j_1} = (v_{i_1}^{j_1}\otimes v_{i_2}^{j_2})\mathbf{G}_1\mathbf{T}_2$$

and

$$(v_{i_1}^{j_1}\otimes v_{i_2}^{j_2})\mathbf{T}_2\mathbf{G}_1 = \xi^{j_2} v_{i_2}^{j_2}\otimes v_{i_1}^{j_1} = (v_{i_1}^{j_1}\otimes v_{i_2}^{j_2})\mathbf{G}_1\mathbf{T}_1$$

and the proof of the Theorem is finished. □

REMARK 2. *Let $\mathcal{K}$ be an R-algebra as in the previous section with corresponding specialized Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}^{\mathcal{K}}(q)$. Then we obtain a specialized tensor product representation $\rho^{\mathcal{K}}:\mathcal{Y}_{r,n}^{\mathcal{K}}(q)\to \mathrm{End}_{\mathcal{K}}(V^{\otimes n})$. Indeed, the above proof amounts only to checking relations, and so carries over to $\mathcal{Y}_{r,n}^{\mathcal{K}}(q)$.*

THEOREM 2.2. *ρ and $\rho^{\mathcal{K}}$ are faithful representations.*

PROOF. We first consider the faithfulness of ρ. Recall Juyumaya's R-basis for $\mathcal{Y}_{r,n}(q)$

$$\mathcal{B}_{r,n} = \{g_\sigma t_1^{j_1}\cdots t_n^{j_n} \mid \sigma\in\mathfrak{S}_n,\ j_i\in\mathbb{Z}/r\mathbb{Z}\}.$$

For $\sigma = s_{i_1}\ldots s_{i_m}\in\mathfrak{S}_n$ written in reduced form we define $\mathbf{G}_\sigma := \mathbf{G}_{i_1}\ldots\mathbf{G}_{i_m}$. To prove that ρ is faithful it is enough to show that

$$\rho(\mathcal{B}_{r,n}) = \{\mathbf{G}_\sigma\mathbf{T}_1^{j_1}\cdots\mathbf{T}_n^{j_n} \mid \sigma\in\mathfrak{S}_n,\ j_k\in\mathbb{Z}/r\mathbb{Z}\}$$

is an R-linearly independent subset of $\mathrm{End}(V^{\otimes n})$. Suppose therefore that there exists a non-trivial linear dependence

$$\sum_{\substack{\sigma\in\mathfrak{S}_n\\ j_i\in\mathbb{Z}/r\mathbb{Z}}} \lambda_{j_1,\ldots,j_n,\sigma}\mathbf{G}_\sigma\mathbf{T}_1^{j_1}\cdots\mathbf{T}_n^{j_n} = 0 \tag{2.6}$$

where not every $\lambda_{j_1,\ldots,j_n,\sigma}\in R$ is zero.

We first observe that for arbitrary a_i's and $\sigma\in\mathfrak{S}_n$ the action of $\mathbf{G}_\sigma$ on the special tensor $v_n^{a_n}\otimes\cdots\otimes v_1^{a_1}$, having the lower indices strictly decreasing, is particularly simple. Indeed, since $\sigma = s_{i_1}\ldots s_{i_m}$ is a reduced expression for σ we have that the action of $\mathbf{G}_\sigma = \mathbf{G}_{i_1}\ldots\mathbf{G}_{i_m}$ in that case always involves the third case of (2.2) and thus is given by place permutation, in other words

$$(v_n^{a_n}\otimes\cdots\otimes v_1^{a_1})\mathbf{G}_\sigma = (v_n^{a_n}\otimes\cdots\otimes v_1^{a_1})\sigma = v_{i_n}^{a_{i_n}}\otimes\cdots\otimes v_{i_1}^{a_{i_1}} \tag{2.7}$$

for some permutation $i_n, \dots, i_1$ of $n, \dots, 1$ uniquely given by σ. Let $\mathfrak{T}_n$ be the R-subalgebra of $\mathrm{End}(V^{\otimes n})$ generated by the $\mathbf{T}_i$'s. For fixed $k_1, \dots, k_n$ we now define

$$V_{k_1,\dots,k_n} := \mathrm{Span}_R\{v_{k_1}^{j_1} \otimes \cdots \otimes v_{k_n}^{j_n} \mid j_k \in \mathbb{Z}/r\mathbb{Z}\}.$$

Then $V_{k_1,\dots,k_n}$ is a $\mathfrak{T}_n$-submodule of $V^{\otimes n}$. Given (2.7), to prove that the linear dependence (2.6) does not exist, it is now enough to show that $V_{k_1,\dots,k_n}$ is a faithful $\mathfrak{T}_n$-module.

For $j = 0, 1, \dots, r-1$ we define $w_k^j \in V$ via

$$w_k^j := \sum_{i=0}^{r-1} \xi^{ij} v_k^i.$$

Then $\{w_k^i \mid i = 0, 1, \dots, r-1, k = 1, \dots, n\}$ is also an R-basis for V, since for fixed k the base change matrix between $\{v_k^i \mid i = 0, 1, \dots, r-1\}$ and $\{w_k^j \mid j = 0, 1, \dots, r-1\}$ is given by a Vandermonde matrix with determinant $\prod_{0 \le i < j \le r-1}(\xi^i - \xi^j)$ which is a unit in R. But then also $\{w_{k_1}^{j_1} \otimes \ldots \otimes w_{k_n}^{j_n} \mid j_i \in \mathbb{Z}/r\mathbb{Z}\}$ is an R-basis for $V_{k_1,\dots,k_n}$. On the other hand, for all j we have that $w_k^j \mathbf{T} = w_k^{j+1}$ where the indices are understood modulo r. Hence, given the nontrivial linear combination in $\mathfrak{T}_n$

$$\sum_{j_i \in \mathbb{Z}/r\mathbb{Z}} \lambda_{j_1,\dots,j_n} \mathbf{T}_1^{j_1} \cdots \mathbf{T}_n^{j_n}$$

we get by acting with it on $w_{k_1}^0 \otimes \ldots \otimes w_{k_n}^0$ the following nonzero element

$$\sum_{j_i \in \mathbb{Z}/r\mathbb{Z}} \lambda_{j_1,\dots,j_n} w_{k_1}^{j_1} \otimes \ldots \otimes w_{k_n}^{j_n}.$$

This proves the Theorem in the case of ρ. The case $\rho^{\mathcal{K}}$ is proved similarly, using that $\prod_{0 \le i < j \le r-1}(\xi^i - \xi^j)$ is a unit in $\mathcal{K}$ as well. □

2.1. The modified Ariki-Koike algebra. In this subsection we obtain our first main result, showing that the Yokonuma-Hecke algebra is isomorphic to a variation of the Ariki-Koike algebra, called the modified Ariki-Koike algebra $\mathcal{H}_{r,n}$. It was introduced by Shoji. Given the faithful tensor representation of the previous subsection, the proof of this isomorphism Theorem is actually almost trivial, but still we think that it is a surprising result. Indeed, the quadratic relations involving the braid group generators look quite different in the two algebras and as a matter of fact the usual Hecke algebra of type A_{n-1} appears naturally as a subalgebra of the (modified) Ariki-Koike algebra, but only as quotient of the Yokonuma-Hecke algebra.

Let us recall Shoji's definition of the modified Ariki-Koike algebra. He defined it over the ring $R_1 := \mathbb{Z}[q, q^{-1}, u_1, \dots, u_r, \Delta^{-1}]$, where $q, u_1, \dots, u_r$ are indeterminates and $\Delta := \prod_{i>j}(u_i - u_j)$ is the Vandermonde determinant. We here consider the modified Ariki-Koike algebra over the ring R, corresponding to a specialization of Shoji's algebra via the homomorphism $\varphi : R_1 \to R$ given by $u_i \mapsto \xi^i$ and $q \mapsto q$.

Let $\mathbf{A}$ be the square matrix of degree r whose ij-entry is given by $\mathbf{A}_{ij} = \xi^{j(i-1)}$ for $1 \le i, j \le r$, i.e. $\mathbf{A}$ is the usual Vandermonde matrix. Then we can write the inverse of $\mathbf{A}$ as $\mathbf{A}^{-1} =$

$\Delta^{-1}\mathbf{B}$, where $\Delta=\prod_{i>j}(\xi^i-\xi^j)$ and $\mathbf{B}=(h_{ij})$ is the adjoint matrix of $\mathbf{A}$, and for $1\le i\le r$ define a polynomial $F_i(X)\in\mathbb{Z}[\xi][X]\subseteq R[X]$ by

$$F_i(X):=\sum_{1\le i\le r}h_{ij}X^{j-1}.$$

DEFINITION 2.3. *The modified Ariki-Koike algebra, denoted $\mathcal{H}_{r,n}=\mathcal{H}_{r,n}(q)$, is the associative R-algebra generated by the elements $h_2,\dots,h_n$ and $\omega_1,\dots,\omega_n$ subject to the following relations:*

$$(h_i-q)(h_i+q^{-1})=0 \qquad \textit{for all } i \tag{2.8}$$

$$h_ih_j=h_jh_i \qquad \textit{for } |i-j|>1 \tag{2.9}$$

$$h_ih_{i+1}h_i=h_{i+1}h_ih_{i+1} \qquad \textit{for all } i=1,\dots,n-2 \tag{2.10}$$

$$(\omega_i-\xi^1)\cdots(\omega_i-\xi^r)=0 \qquad \textit{for all } i \tag{2.11}$$

$$\omega_i\omega_j=\omega_j\omega_i \qquad \textit{for all } i,j \tag{2.12}$$

$$h_j\omega_j=\omega_{j-1}h_j+\Delta^{-2}\sum_{c_1<c_2}(\xi^{c_2}-\xi^{c_1})(q-q^{-1})F_{c_1}(\omega_{j-1})F_{c_2}(\omega_j) \tag{2.13}$$

$$h_j\omega_{j-1}=\omega_jh_j-\Delta^{-2}\sum_{c_1<c_2}(\xi^{c_2}-\xi^{c_1})(q-q^{-1})F_{c_1}(\omega_{j-1})F_{c_2}(\omega_j) \tag{2.14}$$

$$h_j\omega_l=\omega_lh_j \qquad l\neq j,j-1 \tag{2.15}$$

$\mathcal{H}_{r,n}(q)$ was introduced as a way of approximating the usual Ariki-Koike algebra and is isomorphic to it if a certain separation condition holds. In general the two algebras are not isomorphic, but related via a, somewhat mysterious, homomorphism from the Ariki-Koike algebra to $\mathcal{H}_{r,n}(q)$, see [**42**].

Sakamoto and Shoji, [**42**] and [**41**], gave a $\mathcal{H}_{r,n}(q)$-module structure on $V^{\otimes n}$ that we now explain. We first introduce a total order on the v_i^j's via

$$v_1^1,v_2^1,\dots,v_n^1,v_1^2,\dots,v_n^2,\dots,v_1^r,\dots,v_n^r \tag{2.16}$$

and denote by $v_1,\dots,v_{rn}$ these vectors in this order. We then define the linear operator $\mathbf{H}\in\mathrm{End}(V^{\otimes 2})$ as follows:

$$(v_i\otimes v_j)\mathbf{H}:=\begin{cases} qv_i\otimes v_j & \text{if } i=j\\ v_j\otimes v_i & \text{if } i>j\\ (q-q^{-1})v_i\otimes v_j+v_j\otimes v_i & \text{if } i<j.\end{cases}$$

We then extend this to an operator $\mathbf{H}_i$ of $V^{\otimes n}$ by letting $\mathbf{H}$ act in the i'th and $i+1$'st factors. This is essentially Jimbo's original operator for constructing tensor representations for the usual Iwahori-Hecke algebra $\mathcal{H}_n$ of type A. The following result is shown in [**42**].

THEOREM 2.3. *The map $\tilde{\rho}:\mathcal{H}_{r,n}(q)\to\mathrm{End}(V^{\otimes n})$ given by $h_j\to\mathbf{H}_i$, $\omega_j\to\mathbf{T}_j$ defines a faithful representation of $\mathcal{H}_{r,n}(q)$.*

We are now in position to prove the following main Theorem.

THEOREM 2.4. *The Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}(q)$ is isomorphic to the modified Ariki-Koike algebra $\mathcal{H}_{r,n}(q)$.*

PROOF. By the previous Theorem and Theorem 2.2 we can identify $\mathcal{Y}_{r,n}(q)$ and $\mathcal{H}_{r,n}(q)$ with the subalgebras $\rho(\mathcal{Y}_{r,n}(q))$ and $\tilde{\rho}(\mathcal{H}_{r,n}(q))$ of $\mathrm{End}(V^{\otimes n})$, respectively. Hence, in order to prove the Theorem we must show that $\rho(\mathcal{Y}_{r,n}(q)) = \tilde{\rho}(\mathcal{H}_{r,n}(q))$. But by definition, we surely have that the $\mathbf{T}_i$'s belong to both subalgebras, since $\mathbf{T}_i = \rho(t_i)$ and $\mathbf{T}_i = \tilde{\rho}(\omega_i)$.

It is therefore enough to show that the $\mathbf{G}_i$'s from $\rho(\mathcal{Y}_{r,n}(q))$ belong to $\tilde{\rho}(\mathcal{H}_{r,n})$, and that the $\mathbf{H}_i$'s from $\tilde{\rho}(\mathcal{H}_{r,n})$ belong to $\rho(\mathcal{Y}_{r,n}(q))$.

On the other hand, the operator $\mathbf{G}$ coincides with the operator denoted by S in Shoji's paper [**42**]. But then Lemma 3.5 of that paper is the equality

$$\mathbf{G}_{i-1} = \mathbf{H}_i - \Delta^{-2}(q-q^{-1}) \sum_{c_1<c_2} F_{c_1}(\mathbf{T}_{i-1}) F_{c_2}(\mathbf{T}_i).$$

Thus, since $\Delta^{-2}(q-q^{-1})\sum_{c_1<c_2} F_{c_1}(\mathbf{T}_{i-1})F_{c_2}(\mathbf{T}_i)$ belongs to both algebras $\tilde{\rho}(\mathcal{H}_{r,n}(q))$ and $\rho(\mathcal{Y}_{r,n}(q))$, the Theorem follows. □

Lusztig gave in [**30**] a structure Theorem for $\mathcal{Y}_{r,n}(q)$, showing that it is a direct sum of matrix algebras over Iwahori-Hecke algebras of type A. This result was recently recovered by Jacon and Poulain d'Andecy in [**21**]. We now briefly explain how this result, via our isomorphism Theorem, is equivalent to a similar result for $\mathcal{H}_{r,n}(q)$, obtained in [**20**] and [**42**]

For a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ of n of length r, we let $\mathcal{H}_\mu(q)$ be the corresponding *Young-Hecke algebra*, by which we mean that $\mathcal{H}_\mu(q)$ is the R-subalgebra of $\mathcal{H}_n(q)$ generated by the h_i's for $i \in \Sigma_n \cap \mathfrak{S}_\mu$. Thus $\mathcal{H}_\mu(q) \cong \mathcal{H}_{\mu_1}(q) \otimes \ldots \otimes \mathcal{H}_{\mu_r}(q)$ where each factor $\mathcal{H}_{\mu_i}(q)$ is a Iwahori-Hecke algebra corresponding to the indices given by the part μ_i. Let p_μ denote the multinomial coefficient

$$p_\mu := \binom{n}{\mu_1 \cdots \mu_r}. \tag{2.17}$$

With this notation, the structure Theorem due to Lusztig and Jacon-Poulain d'Andecy is as follows

$$\mathcal{Y}_{r,n}(q) \cong \bigoplus_{\mu=(\mu_1,\mu_2,\ldots,\mu_r)\models n} \mathrm{Mat}_{p_\mu}(\mathcal{H}_\mu(q)) \tag{2.18}$$

where for any R-algebra $\mathcal{A}$, we denote by $\mathrm{Mat}_m(\mathcal{A})$ the $m \times m$ matrix algebra with entries in $\mathcal{A}$.

On the other hand, a similar structure Theorem was established for the modified Ariki-Koike algebra $\mathcal{H}_{r,n}(q)$, independently by Sawada and Shoji in [**41**] and by Hu and Stoll in [**20**]:

$$\mathcal{H}_{r,n}(q) \cong \bigoplus_{\mu=(\mu_1,\mu_2,\ldots,\mu_r)\models n} \mathrm{Mat}_{p_\mu}(\mathcal{H}_\mu(q)). \tag{2.19}$$

Thus, our isomorphism Theorem 2.4 shows that above two structure Theorems are equivalent.

We finish this section by showing the following embedding Theorem, already announced above. It is also a consequence of our tensor space module for $\mathcal{Y}_{r,n}(q)$.

THEOREM 2.5. *Suppose that $r \geq n$. Then the homomorphism $\varphi : \mathcal{E}_n^{\mathcal{K}}(q) \to \mathcal{Y}_{r,n}^{\mathcal{K}}(q)$ introduced in Lemma 16 is an embedding.*

In order to prove Theorem 3.1 we need to modify the proof of Corollary 4 of [**39**] to make it valid for general $\mathcal{K}$. For this we first prove the following Lemma.

LEMMA 3. *Let $\mathcal{K}$ be an R-algebra as above and let $A = (I_1, \dots, I_d) \in \mathcal{SP}_n$ be a set partition. Denote by V_A the $\mathcal{K}$-submodule of $V^{\otimes n}$ spanned by the vectors*

$$v_n^{j_n} \otimes \cdots \otimes v_k^{j_k} \otimes \cdots \otimes v_l^{j_l} \otimes \cdots \otimes v_1^{j_1} \qquad 0 \leq j_i \leq r-1$$

with decreasing lower indices and satisfying that $j_k = j_l$ exactly if k and l belong to the same block I_i of A. Let $E_A \in \mathcal{E}_n^{\mathcal{K}}(q)$ be the element defined the same way as $E_A \in \mathcal{Y}_{r,n}(q)$, that is via formula (1.16). Then for all $v \in V_A$ we have that $vE_A = v$ whereas $vE_B = 0$ for $B \in \mathcal{SP}_n$ satisfying $B \not\subseteq A$ with respect to the order $\subseteq$ introduced above.

PROOF. In order to prove the first statement it is enough to show that e_{kl} acts as the identity on the basis vectors of V_A whenever k and l belong to the same block of A. But this follows from the expression for e_{kl} given in (1.11) together with the definition (2.2) of the action of $\mathbf{G}_i$ on $V^{\otimes n}$ and Lemma 2. Just as in the proof of Theorem 2.2 we use that the action of $\mathbf{G}_i$ on $v \in V_A$ is just permutation of the i'th and $i+1$'st factors of v since the lower indices are decreasing.

In order to show the second statement, we first remark that the condition $B \not\subseteq A$ means that there exist k and l belonging to the same block of B, but to different blocks of A. In other words e_{kl} appears as a factor of the product defining E_B whereas for all basis vectors of V_A

$$v_n^{j_n} \otimes \cdots \otimes v_k^{j_k} \otimes \cdots \otimes v_l^{j_l} \otimes \cdots \otimes v_1^{j_1}$$

we have that $j_k \neq j_l$. Just as above, using that the action of $\mathbf{G}_i$ is given by place permutation when the lower indices are decreasing, we deduce from this that $V_A e_{kl} = 0$ and so finally that $V_A E_B = 0$, as claimed. □

PROOF OF THEOREM 3.1. It is enough to show that the composition $\rho^{\mathcal{K}} \circ \varphi$ is injective since we know from Theorem 2.2 that $\rho^{\mathcal{K}}$ is faithful. Now recall from Theorem 2 of [**39**] that the set $\{E_A g_w | A \in \mathcal{SP}_n, w \in \mathfrak{S}_n\}$ generates $\mathcal{E}_n(q)$ over $\mathbb{C}[q, q^{-1}]$ (it is even a basis). The proof of this does not involve any special properties of $\mathbb{C}$ and hence $\{E_A g_w | A \in \mathcal{SP}_n, w \in \mathfrak{S}_n\}$ also generates $\mathcal{E}_n^{\mathcal{K}}(q)$ over $\mathcal{K}$.

Let us now consider a nonzero element $\sum_{w,A} r_{w,A} E_A G_w$ in $\mathcal{E}_n^{\mathcal{K}}(q)$. Under $\rho^{\mathcal{K}} \circ \varphi$ it is mapped to $\sum_{w,A} r_{w,A} \mathbf{E}_A \mathbf{G}_w$ which we must show to be nonzero.

For this we choose $A_0 \in \mathcal{SP}_n$ satisfying $r_{w,A_0} \neq 0$ for some $w \in \mathfrak{S}_n$ and minimal with respect to this under our order $\subseteq$ on $\mathcal{SP}_n$. Let $v \in V_{A_0} \setminus \{0\}$ where V_{A_0} is defined as in the previous Lemma 17. Note that the condition $r \geq n$ ensures that $V_{A_0} \neq 0$, so such a v does exist. Then the Lemma gives us that

$$v\Big(\sum_{w,A} r_{w,A}\mathbf{E}_A\mathbf{G}_w\Big) = v\Big(\sum_{w} r_{w,A_0}\mathbf{G}_w\Big). \tag{2.20}$$

The lower indices of v are strictly decreasing and so each $\mathbf{G}_w$ acts on it by place permutation. It follows from this that (1.11) is nonzero, and the Theorem is proved. □

REMARK 3. *The above proof did not use the linear independence of* $\{E_A g_w \mid A \in \mathcal{SP}_n,$ $w \in \mathfrak{S}_n\}$ *over* $\mathcal{K}$. *In fact, it gives a new proof of Corollary 4 of* [**39**].

In the special case $\mathcal{K} = \mathbb{C}[q,q^{-1}]$ *and* $r = n$ *the Theorem is an immediate consequence of the faithfulness of the tensor product* $V^{\otimes n}$ *as an* $\mathcal{E}_n(q)$*-module, as proved in Corollary 4 of* [**39**]. *Indeed, let* $\rho_{\mathcal{E}_n}^{\mathbb{C}[q,q^{-1}]} : \mathcal{E}_n(q) \to \mathrm{End}(V^{\otimes n})$ *be the homomorphism associated with the* $\mathcal{E}_n(q)$*-module structure on* $V^{\otimes n}$, *introduced in* [**39**]. *Then the injectivity of* $\rho_{\mathcal{E}_n}^{\mathbb{C}[q,q^{-1}]}$ *together with the factorization* $\rho_{\mathcal{E}_n}^{\mathbb{C}[q,q^{-1}]} = \rho^{\mathbb{C}[q,q^{-1}]} \circ \varphi_{\mathbb{C}[q,q^{-1}]}$ *shows directly that* $\varphi_{\mathbb{C}[q,q^{-1}]}$ *is injective. One actually checks that the proof of Corollary 4 of* [**39**] *remains valid for* $\mathcal{K} = R$ *and* $r \geq n$, *but still this is not enough to prove injectivity of* $\varphi = \varphi_{\mathcal{K}}$ *for a general* $\mathcal{K}$ *since extension of scalars from* R *to* $\mathcal{K}$ *is not left exact. Note that the specialization argument of* [**39**] *fails for general* $\mathcal{K}$.

3. Cellular basis for the Yokonuma-Hecke algebra

The goal of this section is to construct a cellular basis for the Yokonuma-Hecke algebra. The cellularity of the Yokonuma-Hecke algebra could also have been obtained from the cellularity of the modified Ariki-Koike algebra, see [**41**], via our isomorphism Theorem from the previous section. We have several reasons for still giving a direct construction of a cellular basis for the Yokonuma-Hecke algebra. Firstly, we believe that our construction is simpler and more natural than the one in [**41**]. Secondly, our basis turns out to have a nice compatiblity property with the subalgebra $\mathfrak{T}_n$ of $\mathcal{Y}_{r,n}(q)$ studied above, a compatibility that we would like to emphasize. This compatibiliy is essential for our proof of Lusztig's presentation for $\mathcal{Y}_{r,n}(q)$, given at the end of this section. We also need the cellular basis in order to show, in the following section, that the Jucys-Murphy operators introduced by Chlouveraki and Poulain d'Andecy are JM-elements in the abstract sense introduced by Mathas. Finally, several of the methods for the construction of the basis are needed in the last section where the algebra of braids and ties is treated.

For our cellular basis for $\mathcal{Y}_{r,n}(q)$ we use for Λ the set $Par_{r,n}$ of r-multipartitions of n, endowed with the dominance order as explained in Preliminars and for $T(\lambda)$ we use the set of standard r-multitableaux $\mathrm{Std}(\boldsymbol{\lambda})$, introduced in the same part. For $* : \mathcal{Y}_{r,n}(q) \to \mathcal{Y}_{r,n}(q)$ we use the R-linear antiautomorphism of $\mathcal{Y}_{r,n}(q)$ determined by $g_i^* = g_i$ and $t_k^* = t_k$ for $1 \leq i < n$ and $1 \leq k \leq n$. Note that $*$ does exist as can easily be checked from the relations defining $\mathcal{Y}_{r,n}(q)$.

We then only have to explain the construction of the basis element itself, for pairs of standard tableaux. Our guideline for this is Murphy's construction of *the standard basis* of the Iwahori-Hecke algebra $\mathcal{H}_n(q)$.

For $\boldsymbol{\lambda} \in Comp_{r,n}$ we first define

$$x_{\boldsymbol{\lambda}} := \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} q^{\ell(w)} g_w \in \mathcal{Y}_{r,n}(q). \tag{3.1}$$

In the case of the Iwahori-Hecke algebra $\mathcal{H}_n(q)$, and $\boldsymbol{\lambda}$ a usual composition, the element $x_{\boldsymbol{\lambda}}$ is the starting point of Murphy's standard basis, corresponding to the most dominant tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$.

In our $\mathcal{Y}_{r,n}(q)$ case, the element $x_{\boldsymbol{\lambda}}$ will only be the first ingredient of the cellular basis element corresponding to the tableau $\mathbf{t}^{\boldsymbol{\lambda}}$. Let us now explain the other two ingredients.

For a composition $\mu = (\mu_1, \dots, \mu_k)$ we define the *reduced composition* $\operatorname{red}\mu$ as the composition obtained from μ by deleting all zero parts $\mu_i = 0$ from μ. We say that a composition μ is reduced if $\mu = \operatorname{red}\mu$.

For any reduced composition $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ we introduce the set partition $A_\mu := (I_1, I_2, \dots, I_k)$ by filling in the numbers consecutively, that is

$$I_1 := \{1, 2, \dots, \mu_1\},\ I_2 := \{\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2\},\ etc. \tag{3.2}$$

and for a multicomposition $\boldsymbol{\lambda} \in Comp_{r,n}$ we define $A_{\boldsymbol{\lambda}} := A_{\operatorname{red}\|\boldsymbol{\lambda}\|} \in \mathcal{SP}_n$. Thus we get for any $\boldsymbol{\lambda} \in Comp_{r,n}$ an idempotent $E_{A_{\boldsymbol{\lambda}}} \in \mathcal{Y}_{r,n}(q)$ which will be the second ingredient of our $\mathcal{Y}_{r,n}(q)$-element for $\mathbf{t}^{\boldsymbol{\lambda}}$. We shall from now on use the notation

$$E_{\boldsymbol{\lambda}} := E_{A_{\boldsymbol{\lambda}}}. \tag{3.3}$$

Clearly $t_i E_{\boldsymbol{\lambda}} = E_{\boldsymbol{\lambda}} t_i$ for all i. Moreover $E_{\boldsymbol{\lambda}}$ satisfies the following key property.

LEMMA 4. *Let $\boldsymbol{\lambda} \in Comp_{r,n}$ and let $A_{\boldsymbol{\lambda}}$ be the associated set partition. Suppose that k and l belong to the same block of $A_{\boldsymbol{\lambda}}$. Then $t_k E_{\boldsymbol{\lambda}} = t_l E_{\boldsymbol{\lambda}}$.*

PROOF. This follows from the definitions. □

From Juyumaya's basis (1.9) it follows that t_i is a diagonalizable element on $\mathcal{Y}_{r,n}(q)$. The eigenspace projector for the action t_i on $\mathcal{Y}_{r,n}(q)$ with eigenvalue ξ^k is

$$u_{ik} = \frac{1}{r} \sum_{j=0}^{r-1} \xi^{-jk} t_i^j \in \mathcal{Y}_{r,n}(q) \tag{3.4}$$

that is $\{v \in \mathcal{Y}_{r,n}(q) \mid t_i v = \xi^k v\} = u_{ik} \mathcal{Y}_{r,n}(q)$.

For $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in Comp_{r,n}$ we define $U_{\boldsymbol{\lambda}}$ as the product

$$U_{\boldsymbol{\lambda}} := \prod_{j=1}^{r} u_{i_j, j} \tag{3.5}$$

where i_j is any number from the j'th component $\mathfrak{t}^{\lambda^{(j)}}$ of $\mathbf{t}^{\boldsymbol{\lambda}}$ with the convention that $u_{i_j,j} := 1$ if it is empty.

We have now gathered all the ingredients of our cellular basis element corresponding to $\mathfrak{t}^{\boldsymbol{\lambda}}$.

DEFINITION 2.4. *Let $\boldsymbol{\lambda} \in Comp_{r,n}$. Then we define $m_{\boldsymbol{\lambda}} \in \mathcal{Y}_{r,n}(q)$ via*

$$m_{\boldsymbol{\lambda}} := U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}}. \tag{3.6}$$

The following Lemmas contain some basic properties for $m_{\boldsymbol{\lambda}}$.

LEMMA 5. *The following properties for $m_{\boldsymbol{\lambda}}$ are true.*

(1) *$U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}}$ is an idempotent. It is independent of the choices of i_j's and so also $m_{\boldsymbol{\lambda}}$ is independent of the choices of i_j's.*
(2) *For i in the j'th component of $\mathfrak{t}^{\boldsymbol{\lambda}}$ (that is $p_{\boldsymbol{\lambda}}(i) = j$) we have $t_i m_{\boldsymbol{\lambda}} = m_{\boldsymbol{\lambda}} t_i = \xi^j m_{\boldsymbol{\lambda}}$.*
(3) *The factors $U_{\boldsymbol{\lambda}}$, $E_{\boldsymbol{\lambda}}$ and $x_{\boldsymbol{\lambda}}$ of $m_{\boldsymbol{\lambda}}$ commute with each other.*
(4) *If i and j occur in the same block of $A_{\boldsymbol{\lambda}}$ then $m_{\boldsymbol{\lambda}} e_{ij} = e_{ij} m_{\boldsymbol{\lambda}} = m_{\boldsymbol{\lambda}}$.*
(5) *If i and j occur in two different blocks of $A_{\boldsymbol{\lambda}}$ then $m_{\boldsymbol{\lambda}} e_{ij} = 0 = e_{ij} m_{\boldsymbol{\lambda}}$.*
(6) *For all $w \in \mathfrak{S}_{\boldsymbol{\lambda}}$ we have $m_{\boldsymbol{\lambda}} g_w = g_w m_{\boldsymbol{\lambda}} = q^{\ell(w)} m_{\boldsymbol{\lambda}}$.*

PROOF. The properties (1) and (2) are consequences of the definitions, whereas (3) follows from (2) and Lemma 1. The property (4) follows from (2) and (3) since $U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} e_{ij} = U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}}$ in that case. Similarly, under the hypothesis of (5) we have that $U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} e_{ij} = 0$ and so also (5) follows from (2) and (3). To show (6) we note that for $s_i \in \mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ we have that $E_{\boldsymbol{\lambda}} g_i^2 = E_{\boldsymbol{\lambda}}(1 + (q - q^{-1}) g_i)$. Since $\mathfrak{S}_{\boldsymbol{\lambda}}$ is a subgroup of $\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ the statement of (6) reduces to the similar Iwahori-Hecke algebra statement for $x_{\boldsymbol{\lambda}}$ which is proved for instance in [**35**, Lemma 3.2]. □

REMARK 4. *Note that i and j are in the same block of $\mathcal{A}_{\boldsymbol{\lambda}}$ if and only if they are in the same component of $\mathfrak{t}^{\boldsymbol{\lambda}}$. However, the enumerations of the blocks of $\mathcal{A}_{\boldsymbol{\lambda}}$ and the components of $\mathfrak{t}^{\boldsymbol{\lambda}}$ are different since $\mathfrak{t}^{\boldsymbol{\lambda}}$ may have empty components and so in part (2) of the Lemma we cannot replace one by the other.*

LEMMA 6. *Let $\boldsymbol{\lambda} \in Comp_{r,n}$ and suppose that $w \in \mathfrak{S}_n$. Then $m_{\boldsymbol{\lambda}} g_w g_i =$*

$$\begin{cases} m_{\boldsymbol{\lambda}} g_{ws_i} \text{ if } \ell(ws_i) > \ell(w) \\ m_{\boldsymbol{\lambda}} g_{ws_i} \text{ if } \ell(ws_i) < \ell(w) \text{ and } i, i+1 \text{ are in different blocks of } (A_{\boldsymbol{\lambda}})w \\ m_{\boldsymbol{\lambda}}(g_{ws_i} + (q - q^{-1}) g_w) \text{ if } \ell(ws_i) < \ell(w) \text{ and } i, i+1 \text{ are in the same block of } (A_{\boldsymbol{\lambda}})w. \end{cases}$$

PROOF. Suppose that $\ell(ws_i) > \ell(w)$ and let $s_{j_1} \cdots s_{j_k}$ be a reduced expression for w. Then $s_{j_1} \cdots s_{j_k} s_i$ is a reduced expression for ws_i and so $g_{ws_i} = g_w g_{s_i}$ by definition. On the other hand, if $\ell(ws_i) < \ell(w)$ then w has a reduced expression ending in s_i, therefore

$$g_w g_i = g_{ws_i} g_i^2 = g_{ws_i}(1 + (q - q^{-1}) e_i g_i) = g_{ws_i} + (q - q^{-1}) g_w e_i.$$

On the other hand, from Lemma 1 we have that $E_{\boldsymbol{\lambda}} g_w e_i = g_w E_{A_{\boldsymbol{\lambda}} w} e_i$ which is equal to $g_w E_{A_{\boldsymbol{\lambda}} w}$ or zero depending on whether i and $i+1$ are in the same block of $A_{\boldsymbol{\lambda}}$ or not. This concludes the proof of the Lemma □

With these preparations, we are in position to give the definition of the set of elements that turn out to contain the cellular basis for $\mathcal{Y}_{r,n}(q)$.

DEFINITION 2.5. *Let $\boldsymbol{\lambda} \in Comp_{r,n}$ and suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard multi-tableaux of shape $\boldsymbol{\lambda}$. Then we define*

$$m_{\mathfrak{s}\mathfrak{t}} := g^*_{d(\mathfrak{s})} m_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})}. \tag{3.7}$$

In particular we have $m_{\boldsymbol{\lambda}} = m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$.

Recall that Murphy introduced the elements $x_{\mathfrak{s}\mathfrak{t}}$ of the Iwahori-Hecke algebra $\mathcal{H}_n(q)$, via

$$x_{\mathfrak{s}\mathfrak{t}} = h^*_{d(\mathfrak{s})} x_\lambda h_{d(\mathfrak{t})} \tag{3.8}$$

for $\mathfrak{s}, \mathfrak{t}$ row standard λ-tableaux. We consider our elements $m_{\mathfrak{s}\mathfrak{t}}$ as the natural generalization of these $x_{\mathfrak{s}\mathfrak{t}}$ to the Yokonuma-Hecke algebra.

Clearly we have $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$, as one sees from the definition of $*$.

Let $\boldsymbol{\lambda} \in Comp_{r,n}$ and set $\alpha := \|\boldsymbol{\lambda}\|$. We have a canonical decomposition of the corresponding Young subgroup

$$\mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \ldots \times \mathfrak{S}_{\alpha_r} \tag{3.9}$$

where $\mathfrak{S}_{\alpha_1}$ is the subgroup of $\mathfrak{S}_n$ permuting $\{1,2,\ldots,\alpha_1\}$, whereas $\mathfrak{S}_{\alpha_2}$ is the subgroup permuting $\{\alpha_1+1, \alpha_1+2, \ldots, \alpha_1+\alpha_2\}$, and so on. Note that this notation deviates slightly from the notation introduced above where $\mathfrak{S}_{\alpha_i}$ is the symmetric group on the numbers $\{1,2,\ldots,\alpha_i\}$; this kind of abuse of notation, that we shall use frequently in the following, should not cause confusion. We now define

$$\mathcal{Y}_\alpha(q) := \mathrm{Span}_R\{U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} g_w \,|\, w \in \mathfrak{S}_\alpha\}. \tag{3.10}$$

We have the following Lemma.

LEMMA 7. *$\mathcal{Y}_\alpha(q)$ is a subalgebra of $\mathcal{Y}_{r,n}(q)$. Its identity element is given by the central idempotent $U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}}$. There is an isomorphism between the Young-Hecke algebra $\mathcal{H}_\alpha(q)$ and $\mathcal{Y}_\alpha(q)$ given by*

$$\mathcal{H}_\alpha(q) \longrightarrow \mathcal{Y}_\alpha(q), \; g_w \mapsto U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} g_w \quad \text{where } w \in \mathfrak{S}_\alpha. \tag{3.11}$$

Using the canonical isomorphism $\mathcal{H}_\alpha(q) \cong \mathcal{H}_{\alpha_1}(q) \otimes \cdots \otimes \mathcal{H}_{\alpha_r}(q)$ it is given by

$$\mathcal{H}_{\alpha_1}(q) \otimes \cdots \otimes \mathcal{H}_{\alpha_r}(q) \longrightarrow \mathcal{Y}_\alpha(q), \; a_1 \otimes \cdots \otimes a_r \mapsto U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}}\, a_1 \cdots a_r \quad \text{where } a_i \in \mathcal{H}_{\alpha_i}(q). \tag{3.12}$$

PROOF. From Lemma 5 we know that $U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}}$ is an idempotent. For $w \in \mathfrak{S}_\alpha$ it commutes with g_w as can be seen by combining the Yokonuma-Hecke algebra relation (1.3) with Lemma 5, and hence it is central. Moreover, for $s_i \in \mathfrak{S}_\alpha$ we have that $E_{\boldsymbol{\lambda}} g_i^2 = E_{\boldsymbol{\lambda}}(1 + (q - q^{-1}) g_i)$, as mentioned in the proof of Lemma 5, and so we also have

$$U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} g_i^2 = U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} (1 + (q - q^{-1}) g_i) \;\text{ for } s_i \in \mathfrak{S}_\alpha.$$

It follows from this that $\mathcal{Y}_\alpha(q)$ is a subalgebra of $\mathcal{Y}_{r,n}(q)$ and that there is a homomorphism from $\mathcal{H}_\alpha(q)$ to $\mathcal{Y}_\alpha(q)$ given by $g_w \mapsto U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} g_w$. On the other hand, it is clearly surjective and using Juyumaya's basis (1.9), we get that $\mathcal{Y}_\alpha(q)$ has the same dimension as $\mathcal{H}_\alpha(q)$ and so the Lemma follows. □

REMARK 5. *Suppose still that $\boldsymbol{\lambda} \in Comp_{r,n}$ with $\alpha := \|\boldsymbol{\lambda}\|$. If $\mathfrak{s}$ and $\mathfrak{t}$ are $\boldsymbol{\lambda}$-multitableaux of the initial kind, we may view $m_{\mathfrak{s}\mathfrak{t}}$ as usual Murphy elements of the Young-Hecke algebra. Indeed, in this case $d(\mathfrak{s})$ and $d(\mathfrak{t})$ belong to $\mathfrak{S}_\alpha$ and hence, using the above Lemma, we have that $g_{d(\mathfrak{s})}$ and $g_{d(\mathfrak{t})}$ commute with $U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}}$. In particular, we have that*

$$m_{\mathfrak{s}\mathfrak{t}} = g^*_{d(\mathfrak{s})} U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})} = U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} g^*_{d(\mathfrak{s})} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})}. \tag{3.13}$$

We have that $\mathfrak{S}_{\boldsymbol{\lambda}}$ is subgroup of $\mathfrak{S}_\alpha$ compatible with (3.9) in the sense that

$$\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\lambda^{(1)}} \times \mathfrak{S}_{\lambda^{(2)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}, \text{ where } \mathfrak{S}_{\lambda^{(i)}} \leq \mathfrak{S}_{\alpha_i}. \tag{3.14}$$

Here $\mathfrak{S}_{\lambda^{(i)}}$ is subject to the same abuse of notation as $\mathfrak{S}_{\alpha_i}$. We then get a corresponding factorization

$$x_{\boldsymbol{\lambda}} = x_{\lambda^{(1)}} x_{\lambda^{(2)}} \cdots x_{\lambda^{(r)}} \tag{3.15}$$

where $x_{\lambda^{(i)}} = \sum_{w \in \mathfrak{S}_{\lambda^{(i)}}} g_w$. Since $\mathfrak{s}$ and $\mathfrak{t}$ are of the initial kind we get decompositions, corresponding to the decomposition in (3.9)

$$d(\mathfrak{s}) = \left(d(\mathfrak{s}^{(1)}), d(\mathfrak{s}^{(2)}), \cdots, d(\mathfrak{s}^{(r)})\right) \quad \text{and} \quad d(\mathfrak{t}) = \left(d(\mathfrak{t}^{(1)}), d(\mathfrak{t}^{(2)}), \cdots, d(\mathfrak{t}^{(r)})\right). \tag{3.16}$$

But then from (3.13) we get a decomposition of $m_{\mathfrak{s}\mathfrak{t}}$ as follows

$$\begin{aligned} m_{\mathfrak{s}\mathfrak{t}} &= U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} g^*_{d(\mathfrak{s}^{(1)})} x_{\lambda^{(1)}} g_{d(\mathfrak{t}^{(1)})} g^*_{d(\mathfrak{s}^{(2)})} x_{\lambda^{(2)}} g_{d(\mathfrak{t}^{(2)})} \cdots g^*_{d(\mathfrak{s}^{(r)})} x_{\lambda^{(r)}} g_{d(\mathfrak{t}^{(r)})} \\ &= U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} x_{\mathfrak{s}^{(1)}\mathfrak{t}^{(1)}} x_{\mathfrak{s}^{(2)}\mathfrak{t}^{(2)}} \cdots x_{\mathfrak{s}^{(r)}\mathfrak{t}^{(r)}} \end{aligned} \tag{3.17}$$

where $x_{\mathfrak{s}^{(i)}\mathfrak{t}^{(i)}} := g^*_{d(\mathfrak{s}^{(i)})} x_{\lambda^{(i)}} g_{d(\mathfrak{t}^{(i)})}$. Under the isomorphism of the Lemma, we then get via (3.17) that $m_{\mathfrak{s}\mathfrak{t}}$ corresponds to

$$x_{\mathfrak{s}^{(1)}\mathfrak{t}^{(1)}} \otimes x_{\mathfrak{s}^{(2)}\mathfrak{t}^{(2)}} \otimes \cdots \otimes x_{\mathfrak{s}^{(r)}\mathfrak{t}^{(r)}} \in \mathcal{H}_\alpha(q) \tag{3.18}$$

where each $x_{\mathfrak{s}^{(i)}\mathfrak{t}^{(i)}} \in \mathcal{H}_{\alpha_i}(q)$ is a usual Murphy element. This explains the claim made in the beginning of the Remark.

Our goal is to show that with $\mathbf{s}$ and $\mathbf{t}$ running over standard multitableaux for multipartitions, the $m_{\mathfrak{s}\mathfrak{t}}$'s form a cellular basis for $\mathcal{Y}_{r,n}(q)$. A first property of $m_{\mathfrak{s}\mathfrak{t}}$ is given by the following Lemma.

LEMMA 8. *Suppose that $\boldsymbol{\lambda} \in Comp_{r,n}$ and that $\mathfrak{s}$ and $\mathfrak{t}$ are $\boldsymbol{\lambda}$-multitableaux. If i and j occur in the same component of $\mathfrak{t}$ then we have that $m_{\mathfrak{s}\mathfrak{t}} e_{ij} = m_{\mathfrak{s}\mathfrak{t}}$. Otherwise $m_{\mathfrak{s}\mathfrak{t}} e_{ij} = 0$. A similar statement holds for $e_{ij} m_{\mathfrak{s}\mathfrak{t}}$.*

PROOF. From the definitions we have and Lemma 1 we have that

$$m_{\mathfrak{s}\mathfrak{t}} e_{ij} = g^*_{d(\mathfrak{s})} x_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} U_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})} e_{ij} = g^*_{d(\mathfrak{s})} m_{\boldsymbol{\lambda}} e_{id(\mathfrak{t})^{-1}, jd(\mathfrak{t})^{-1}} g_{d(\mathfrak{t})}.$$

But i and j occur in the same component of $\mathfrak{t}$ iff $id(\mathfrak{t})^{-1}$ and $jd(\mathfrak{t})^{-1}$ occur in the same block of $A_{\boldsymbol{\lambda}}$ and so the first part of the Lemma follows from (4) and (5) of Lemma 5. The second part is proved similarly or by applying $*$ to the first part. □

LEMMA 9. *Let $\boldsymbol{\lambda} \in Comp_{r,n}$ and let $\mathfrak{s}$ and $\mathfrak{t}$ be row standard $\boldsymbol{\lambda}$-multitableaux. Then for $h \in \mathcal{Y}_{r,n}(q)$ we have that $m_{\mathfrak{s}\mathfrak{t}}h$ is a linear combination of terms of the form $m_{\mathfrak{s}\mathfrak{v}}$ where $\mathfrak{v}$ is a row standard $\boldsymbol{\lambda}$-multitableau. A similar statement holds for $hm_{\mathfrak{s}\mathfrak{t}}$.*

PROOF. Using Lemma 6 we get that $m_{\mathfrak{s}\mathfrak{t}}h$ is a linear combination of terms of the form $m_{\mathfrak{s}\mathfrak{t}^{\lambda}}g_w$. For each such w we find a $y \in \mathfrak{S}_{\boldsymbol{\lambda}}$ and a distinguished right coset representative d of $\mathfrak{S}_{\boldsymbol{\lambda}}$ in $\mathfrak{S}_n$ such that $w = yd$ and $\ell(w) = \ell(y) + \ell(d)$. Hence, via Lemma 6 we get that

$$m_{\mathfrak{s}\mathfrak{t}}h = q^{\ell(y)} m_{\mathfrak{s}\mathfrak{t}^{\lambda}} g_d = q^{\ell(y)} m_{\mathfrak{s}\mathfrak{v}}$$

where $\mathfrak{v} = \mathfrak{t}^{\boldsymbol{\lambda}} g_d$ is row standard. This proves the Lemma in the case $m_{\mathfrak{s}\mathfrak{t}}h$. The case $hm_{\mathfrak{s}\mathfrak{t}}$ is treated similarly or by applying $*$ to the first case.

□

The proof of the next Lemma is inspired by the proof of Proposition 3.18 of Dipper, James and Mathas' paper [**11**], although it should be noted that the basic setup of [**11**] is different from ours. Just like in that paper, our proof relies on Murphy's Theorem 4.18 in [**37**], which is a key ingredient for the construction of the standard basis for $\mathcal{H}_n(q)$.

LEMMA 10. *Suppose that $\boldsymbol{\lambda} \in Comp_{r,n}$ and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard $\boldsymbol{\lambda}$-multi-tableaux. Then there are multipartitions $\boldsymbol{\mu} \in Par_{r,n}$ and standard multitableaux $\mathfrak{u}$ and $\mathfrak{v}$ of shape $\boldsymbol{\mu}$, such that $\mathfrak{u} \trianglerighteq \mathfrak{s}$, $\mathfrak{v} \trianglerighteq \mathfrak{t}$ and such that $m_{\mathfrak{s}\mathfrak{t}}$ is a linear combination of the corresponding elements $m_{\mathfrak{u}\mathfrak{v}}$.*

PROOF. Let α be the composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) := \|\boldsymbol{\lambda}\|$ with corresponding Young subgroup $\mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots \times \mathfrak{S}_{\alpha_r}$ (where some of the factors $\mathfrak{S}_{\alpha_i}$ may be trivial). Then there exist $\boldsymbol{\lambda}$-multitableaux $\mathfrak{s}_0$ and $\mathfrak{t}_0$ of the initial kind together with $w_{\mathfrak{s}}, w_{\mathfrak{t}} \in \mathfrak{S}_n$ such that $d(\mathfrak{s}) = d(\mathfrak{s}_0)w_{\mathfrak{s}}$, $d(\mathfrak{t}) = d(\mathfrak{t}_0)w_{\mathfrak{t}}$ and $\ell(d(\mathfrak{s})) = \ell(d(\mathfrak{s}_0)) + \ell(w_{\mathfrak{s}})$ and $\ell(d(\mathfrak{t})) = \ell(d(\mathfrak{t}_0)) + \ell(w_{\mathfrak{t}})$. Thus, $w_{\mathfrak{s}}$ and $w_{\mathfrak{t}}$ are distinguished right coset representatives for $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$ and using Lemma 6, together with its left action version obtained via $*$, we get that $m_{\mathfrak{s}\mathfrak{t}} = g^*_{w_{\mathfrak{s}}} m_{\mathfrak{s}_0\mathfrak{t}_0} g_{w_{\mathfrak{t}}}$. Let $\mathfrak{s}_0 = (\mathfrak{s}_0^{(1)}, \mathfrak{s}_0^{(2)}, \ldots, \mathfrak{s}_0^{(r)})$ and $\mathfrak{t}_0 = (\mathfrak{t}_0^{(1)}, \mathfrak{t}_0^{(2)}, \ldots, \mathfrak{t}_0^{(r)})$. Then under the isomorphism of Lemma 7 we have that $m_{\mathfrak{s}_0\mathfrak{t}_0}$ corresponds to

$$x_{\mathfrak{s}^{(1)}\mathfrak{t}^{(1)}} \otimes x_{\mathfrak{s}^{(2)}\mathfrak{t}^{(2)}} \otimes \cdots \otimes x_{\mathfrak{s}^{(r)}\mathfrak{t}^{(r)}} \in \mathcal{H}_\alpha(q) \tag{3.19}$$

as explained in Remark 5. On each of the factors $x_{\mathfrak{s}_0^{(i)}\mathfrak{t}_0^{(i)}}$ we now use Murphy's result Theorem 4.18 of [**37**] thus concluding that $x_{\mathfrak{s}_0^{(i)}\mathfrak{t}_0^{(i)}}$ is a linear combination of terms of the form $x_{\mathfrak{u}_0^{(i)}\mathfrak{v}_0^{(i)}}$ where $\mathfrak{u}_0^{(i)}$ and $\mathfrak{v}_0^{(i)}$ are standard $\mu_0^{(i)}$-tableaux on the numbers permuted by $\mathfrak{S}_{\alpha_i}$ and satisfying $\mathfrak{u}_0^{(i)} \trianglerighteq \mathfrak{s}_0^{(i)}$ and $\mathfrak{v}_0^{(i)} \trianglerighteq \mathfrak{t}_0^{(i)}$. Letting $\boldsymbol{\mu} := (\mu_0^{(1)}, \mu_0^{(2)}, \ldots, \mu_0^{(r)})$, $\mathfrak{u}_0 := (\mathfrak{u}_0^{(1)}, \mathfrak{u}_0^{(2)}, \ldots, \mathfrak{u}_0^{(r)})$ and $\mathfrak{v}_0 := (\mathfrak{v}_0^{(1)}, \mathfrak{v}_0^{(2)}, \ldots, \mathfrak{v}_0^{(r)})$ and using the isomorphism of Lemma 7 in the other direction we then get that $m_{\mathfrak{s}_0\mathfrak{t}_0}$ is a linear combination of terms $m_{\mathfrak{u}_0\mathfrak{v}_0}$ where $\mathfrak{u}_0$ and $\mathfrak{v}_0$ are standard $\boldsymbol{\mu}$-multitableaux such that $\mathfrak{u}_0 \trianglerighteq \mathfrak{s}_0$ and $\mathfrak{v}_0 \trianglerighteq \mathfrak{t}_0$. Hence $m_{\mathfrak{s}\mathfrak{t}} = g^*_{w_{\mathfrak{s}}} m_{\mathfrak{s}_0\mathfrak{t}_0} g_{w_{\mathfrak{t}}}$ is a linear combination of terms $g^*_{w_{\mathfrak{s}}} m_{\mathfrak{u}_0\mathfrak{v}_0} g_{w_{\mathfrak{t}}}$. On the other hand, $\mathfrak{u}_0$ and $\mathfrak{v}_0$ are of the initial kind, and so we get $g^*_{w_{\mathfrak{s}}} m_{\mathfrak{u}_0\mathfrak{v}_0} g_{w_{\mathfrak{t}}} = m_{\mathfrak{u}_0 w_{\mathfrak{s}}, \mathfrak{v}_0 w_{\mathfrak{t}}}$ since $w_{\mathfrak{s}}$ and $w_{\mathfrak{t}}$ are distinguished right coset representatives for $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$. This also implies that $\mathfrak{u}_0 w_{\mathfrak{s}} \trianglerighteq \mathfrak{s}_0 w_{\mathfrak{s}} = \mathfrak{s}$ and $\mathfrak{v}_0 w_{\mathfrak{s}} \trianglerighteq \mathfrak{t}_0 w_{\mathfrak{s}} = \mathfrak{t}$ proving the Lemma. □

COROLLARY 2.1. *Suppose that $\boldsymbol{\lambda} \in Comp_{r,n}$ and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard $\boldsymbol{\lambda}$-multitableaux. If $h \in \mathcal{Y}_{r,n}(q)$, then $m_{\mathfrak{s}\mathfrak{t}} h$ is a linear combination of terms of the form $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard $\boldsymbol{\mu}$-multitableaux for some multipartition $\boldsymbol{\mu} \in Par_{r,n}$ and $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and $\mathfrak{v} \trianglerighteq \mathfrak{t}$. A similar statement holds for $hm_{\mathfrak{s}\mathfrak{t}}$.*

PROOF. This is now immediate from the Lemmas 9 and 10. □

So far our construction of the cellular basis has followed the layout used in [**11**], with the appropriate adaptions. But to show that the $m_{\mathfrak{s}\mathfrak{t}}$'s generate $\mathcal{Y}_{r,n}(q)$ we shall deviate from that path. We turn our attention to the R-subalgebra $\mathcal{T}_n$ of $\mathcal{Y}_{r,n}$ generated by $t_1, t_2, \ldots, t_n$. By the faithfulness of $V^{\otimes n}$, it is isomorphic to the subalgebra $\mathfrak{T}_n \subset \mathrm{End}(V^{\otimes n})$ considered above. Our proof that the elements $m_{\mathfrak{s}\mathfrak{t}}$ generate $\mathcal{Y}_{r,n}(q)$ relies on the, maybe surprising, fact that $\mathcal{T}_n$ is compatible with the $\{m_{\mathfrak{s}\mathfrak{t}}\}$, in the sense that the elements $\{m_{\mathfrak{s}\mathfrak{s}}\}$ where $\mathfrak{s}$ is a multitableau corresponding to a one-column multipartition induce a basis for $\mathcal{T}_n$.

As already mentioned, we consider our $m_{\mathfrak{s}\mathfrak{t}}$ as the natural generalization of Murphy's standard basis to $\mathcal{Y}_{r,n}(q)$. It is interesting to note that Murphy's standard basis and its generalization have already before manifested 'good' compatibility properties of the above kind.

Let us first define a one-column r-multipartition to be an element of $Par_{r,n}$ of the form $((1^{c_1}), \ldots, (1^{c_r}))$ and let $Par^1_{r,n}$ be the set of one-column r-multipartitions. Note that there is an obvious bijection between $Par^1_{r,n}$ and the set of usual compositions in r parts. We define

$$\mathrm{Std}^1_{n,r} := \left\{ \mathfrak{s} \,|\, \mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in Par^1_{r,n} \right\}.$$

Note that $\mathrm{Std}^1_{n,r}$ has cardinality r^n as follows from the multinomial formula.

LEMMA 11. *For all $\mathfrak{s} \in \mathrm{Std}^1_{n,r}$, we have that $m_{\mathfrak{s}\mathfrak{s}}$ belongs to $\mathcal{T}_n$.*

PROOF. Let $\mathfrak{s}$ be an element of $\mathrm{Std}^1_{n,r}$. It general, it is useful to think of $d(\mathfrak{s}) \in \mathfrak{S}_n$ as the row reading of $\mathfrak{s}$, that is the element obtained by reading the components of $\mathfrak{s}$ from left to right, and the rows of each component from top to bottom.

We show by induction on $\ell(d(\mathfrak{s}))$ that $m_{\mathfrak{s}\mathfrak{s}}$ belongs to $\mathcal{T}_n$. If $\ell(d(\mathfrak{s})) = 0$ then $x_{\boldsymbol{\lambda}} = 1$ and so $m_{\mathfrak{s}\mathfrak{s}} = U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}}$ that certainly belongs to $\mathcal{T}_n$. Assume that the statement holds for all multitableaux $\mathfrak{s}' \in \mathrm{Std}^1_{n,r}$ such that $\ell(d(\mathfrak{s}')) < \ell(d(\mathfrak{s}))$. Choose i such that i occurs in $\mathfrak{s}$ to the right of $i+1$: such an i exists because $\ell(d(\mathfrak{s})) \neq 0$. Then we can apply the inductive hypothesis to $\mathfrak{s}s_i$, that is $m_{\mathfrak{s}s_i\mathfrak{s}s_i} \in \mathcal{T}_n$. But then

$$m_{\mathfrak{s}\mathfrak{s}} = g^*_{d(\mathfrak{s})} m_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} = g_i m_{\mathfrak{s}s_i\mathfrak{s}s_i} g_i = g_i m_{\mathfrak{s}s_i\mathfrak{s}s_i} (g_i^{-1} + (q - q^{-1}) e_i). \tag{3.20}$$

But $g_i m_{\mathfrak{s}s_i\mathfrak{s}s_i} g_i^{-1}$ certainly belongs to $\mathcal{T}_n$, as one sees from relation (1.3). Finally, from Lemma 8 we get that $m_{\mathfrak{s}s_i\mathfrak{s}s_i} e_i = 0$, thus proving the Lemma. □

LEMMA 12. *Suppose that $\boldsymbol{\lambda} \in Comp_{r,n}$ and let $\mathfrak{s}$ and $\mathfrak{t}$ be $\boldsymbol{\lambda}$-multitableaux. Then for all $k = 1, \dots, n$ we have that*

$$m_{\mathfrak{t}\mathfrak{s}} t_k = \xi^{p_{\mathfrak{s}}(k)} m_{\mathfrak{t}\mathfrak{s}} \quad \textit{and} \quad t_k m_{\mathfrak{t}\mathfrak{s}} = \xi^{p_{\mathfrak{t}}(k)} m_{\mathfrak{t}\mathfrak{s}}.$$

PROOF. From (1.3) we have that $g_w t_k = t_{kw^{-1}} g_w$ for all $w \in \mathfrak{S}_n$. Then, by Lemma 5(2) we have

$$m_{\mathfrak{t}^{\lambda}\mathfrak{s}} t_k = m_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} t_k = m_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} t_k = m_{\boldsymbol{\lambda}} t_{kd(\mathfrak{s})^{-1}} g_{d(\mathfrak{s})} = \xi^{p_{\lambda}(kd(\mathfrak{s})^{-1})} m_{\mathfrak{t}^{\lambda}\mathfrak{s}}.$$

On the other hand, since $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}} d(\mathfrak{s})$ we have that $p_{\boldsymbol{\lambda}}(kd(\mathfrak{s})^{-1}) = p_{\mathfrak{s}}(k)$ and hence $m_{\mathfrak{t}^{\lambda}\mathfrak{s}} t_k = \xi^{p_{\mathfrak{s}}(k)} m_{\mathfrak{t}^{\lambda}\mathfrak{s}}$. Multiplying this equality on the left by $g^*_{d(\mathfrak{t})}$, the proof of the first formula is completed. The second formula is shown similarly or by applying $*$ to the first. $\square$

Our next Proposition shows that the set $\{m_{\mathfrak{s}\mathfrak{s}}\}$, where $\mathfrak{s} \in \mathrm{Std}^1_{n,r}$, forms a basis for $\mathcal{T}_n$, as promised. We already know that $m_{\mathfrak{s}\mathfrak{s}} \in \mathcal{T}_n$ and that the cardinality of $\mathrm{Std}^1_{n,r}$ is r^n which is the dimension of $\mathcal{T}_n$, but even so the result is not completely obvious, since we are working over the ground ring R which is not a field.

PROPOSITION 2.1. $\{m_{\mathfrak{s}\mathfrak{s}} \,|\, \mathfrak{s} \in \mathrm{Std}^1_{n,r}\}$ *is an R-basis for $\mathcal{T}_n$.*

PROOF. Recall that we showed in the proof of Theorem 2.2 that

$$V_{i_1,i_2\dots,i_n} = \mathrm{Span}_R\{v^{j_1}_{i_1} \otimes v^{j_2}_{i_2} \otimes \cdots \otimes v^{j_n}_{i_n} \,|\, j_k \in \mathbb{Z}/r\mathbb{Z}\}$$

is a faithful $\mathcal{T}_n$-module for any fixed, but arbitrary, set of lower indices. Let seq_n be the set of sequences $\underline{i} = (i_1, i_2, \dots, i_n)$ of numbers $1 \le i_j \le n$. Then we have that

$$V^{\otimes n} = \bigoplus_{\underline{i} \in \mathrm{seq}_n} V_{\underline{i}} \tag{3.21}$$

and of course $V^{\otimes n}$ is a faithful $\mathcal{T}_n$-module, too. For $\mathfrak{s} \in \mathrm{Std}^1_{n,r}$ and $\underline{i} \in \mathrm{seq}_n$ we define

$$v^{\mathfrak{s}}_{\underline{i}} := v^{j_1}_{i_1} \otimes v^{j_2}_{i_2} \otimes \ldots \otimes v^{j_n}_{i_n} \in V_{\underline{i}} \tag{3.22}$$

where $(j_1, j_2, \dots j_n) := (p_{\mathfrak{s}}(1), p_{\mathfrak{s}}(2), \dots p_{\mathfrak{s}}(n))$. Then $\{v^{\mathfrak{s}}_{\underline{i}} \,|\, \mathfrak{s} \in \mathrm{Std}^1_{n,r}, \underline{i} \in \mathrm{seq}_n\}$ is an R-basis for $V^{\otimes n}$. We now claim the following formula in $V_{\underline{i}}$:

$$v^{\mathfrak{t}}_{\underline{i}} m_{\mathfrak{s}\mathfrak{s}} = \begin{cases} v^{\mathfrak{t}}_{\underline{i}} & \text{if } \mathfrak{s} = \mathfrak{t} \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

We show it by induction on $\ell(d(\mathfrak{s}))$. If $\ell(d(\mathfrak{s})) = 0$, then $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}}$ where $\boldsymbol{\lambda}$ is the shape of $\mathfrak{s}$. We have $x_{\boldsymbol{\lambda}} = 1$ and so $m_{\mathfrak{s}\mathfrak{s}} = m_{\boldsymbol{\lambda}} = U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}}$. We then get (3.23) directly from the definitions of $U_{\boldsymbol{\lambda}}$ and $E_{\boldsymbol{\lambda}}$ together with Lemma 2.

Let now $\ell(d(\mathfrak{s})) \neq 0$ and assume that (3.23) holds for multitableaux $\mathfrak{s}'$ such that $\ell(d(\mathfrak{s}')) < \ell(d(\mathfrak{s}))$. We choose j such that j occurs in $\mathfrak{s}$ to the right of $j+1$. Using (3.20) we have that $m_{\mathfrak{s}\mathfrak{s}} = g_j m_{\mathfrak{s}s_j \mathfrak{s}s_j} g_j^{-1}$. On the other hand, j and $j+1$ occur in different components of $\mathfrak{s}$ and so by Definition 2.2 of the $\mathcal{Y}_{r,n}(q)$-action in $V^{\otimes n}$ we get that $v^{\mathfrak{s}}_{\underline{i}} g_j^{\pm 1} = v^{\mathfrak{s}s_j}_{\underline{i}s_j}$, corresponding to the first case of (2.2). Hence we get via the inductive hypothesis that

$$v^{\mathfrak{s}}_{\underline{i}} m_{\mathfrak{s}\mathfrak{s}} = v^{\mathfrak{s}}_{\underline{i}} g_j m_{\mathfrak{s}s_j \mathfrak{s}s_j} g_j^{-1} = v^{\mathfrak{s}s_j}_{\underline{i}s_j} m_{\mathfrak{s}s_j \mathfrak{s}s_j} g_j^{-1} = v^{\mathfrak{s}s_j}_{\underline{i}s_j} g_j^{-1} = v^{\mathfrak{s}}_{\underline{i}}$$

which shows the first part of (3.23).

If $\mathfrak{s} \neq \mathfrak{t}$ then we essentially argue the same way. We choose j as before and may apply the inductive hypothesis to $\mathfrak{s}s_j$. We have that $v_{\underline{i}}^{\mathfrak{t}} m_{\mathfrak{s}\mathfrak{s}} = v_{\underline{i}}^{\mathfrak{t}} g_j m_{\mathfrak{s}s_j \mathfrak{s}s_j} g_j^{-1}$ and so need to determine $v_{\underline{i}}^{\mathfrak{t}} g_j$. This is slightly more complicated than in the first case, but using the Definition 2.2 of the $\mathcal{Y}_{r,n}(q)$-action in $V^{\otimes n}$ we get that $v_{\underline{i}}^{\mathfrak{t}} g_j$ is always an R-linear combination of the vectors $v_{\underline{i}s_j}^{\mathfrak{t}s_j}$ and $v_{\underline{i}}^{\mathfrak{t}s_j}$: indeed in the cases $s = t$ of Definition 2.2 we have that $p_{\mathfrak{t}s_j}(s) = p_{\mathfrak{t}}(s)$. But $\mathfrak{s} \neq \mathfrak{t}$ implies that $\mathfrak{s}s_j \neq \mathfrak{t}s_j$ and so we get by the inductive hypothesis that

$$v_{\underline{i}}^{\mathfrak{t}} m_{\mathfrak{s}\mathfrak{s}} = v_{\underline{i}}^{\mathfrak{t}} g_j m_{\mathfrak{s}s_j \mathfrak{s}s_j} g_j^{-1} = 0 \tag{3.24}$$

and (3.23) is proved.

From (3.23) we now deduce that $\sum_{\mathfrak{s}\in \mathrm{Std}^1_{n,r}} v_{\underline{i}}^{\mathfrak{t}} m_{\mathfrak{s}\mathfrak{s}} = v_{\underline{i}}^{\mathfrak{t}}$ for any $\mathfrak{t}$ and $\underline{i}$, and hence

$$\sum_{\mathfrak{s}\in \mathrm{Std}^1_{n,r}} m_{\mathfrak{s}\mathfrak{s}} = 1 \tag{3.25}$$

since $V^{\otimes n}$ is faithful and the $\{v_{\underline{i}}^{\mathfrak{t}}\}$ form a basis for $V^{\otimes n}$. We then get that

$$t_i = t_i 1 = \sum_{\mathfrak{s}\in \mathrm{Std}^1_{n,r}} t_i m_{\mathfrak{s}\mathfrak{s}} = \sum_{\mathfrak{s}\in \mathrm{Std}^1_{n,r}} \xi^{p_{\mathfrak{s}}(i)} m_{\mathfrak{s}\mathfrak{s}} \tag{3.26}$$

and hence, indeed, the set $\{m_{\mathfrak{s}\mathfrak{s}} \,|\, \mathfrak{s} \in \mathrm{Std}^1_{n,r}\}$ generates $\mathcal{T}_n$. On the other hand, the R-independence of $\{m_{\mathfrak{s}\mathfrak{s}}\}$ follows easily from (3.23), via evaluation on the vectors $v_{\underline{i}}^{\mathfrak{t}}$. The Theorem is proved. □

THEOREM 2.6. *The algebra $\mathcal{Y}_{r,n}(q)$ is a free R-module with basis*

$$\mathcal{B}_{r,n} = \left\{ m_{\mathfrak{s}\mathfrak{t}} \,|\, \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for some multipartition } \boldsymbol{\lambda} \text{ of } n \right\}.$$

Moreover, $(\mathcal{B}_{r,n}, Par_{r,n})$ is a cellular basis of $\mathcal{Y}_{r,n}(q)$ in the sense of Definition 1.1.

PROOF. From Proposition 2.1, we have that 1 is an R-linear combination of elements $m_{\mathfrak{s}\mathfrak{s}}$ where $\mathfrak{s}$ are certain standard multitableaux. Thus, via Corollary 2.1 we get that $\mathcal{B}_{r,n}$ spans $\mathcal{Y}_{r,n}(q)$. On the other hand, the cardinality of $\mathcal{B}_{r,n}$ is $r^n n!$ since, for example, $\mathcal{B}_{r,n}$ is the set of tableaux for the Ariki-Koike algebra whose dimension is $r^n n!$. But this implies that $\mathcal{B}_{r,n}$ is an R-basis for $\mathcal{Y}_{r,n}(q)$. Indeed, from Juyumaya's basis we know that $\mathcal{Y}_{r,n}(q)$ has rank $N := r^n n!$ and any surjective homomorphism $f : R^N \mapsto R^N$ splits since R^N is a projective R-module.

The multiplicative property that $\mathcal{B}_{r,n}$ must satisfy in order to be a cellular basis of $\mathcal{Y}_{r,n}(q)$, can now be shown by repeating the argument of Proposition 3.25 of [**11**]. For the reader's convenience, we sketch the argument.

Let first $\widehat{\mathcal{Y}^{\boldsymbol{\lambda}}_{r,n}}(q)$ be the R-submodule of $\mathcal{Y}_{r,n}(q)$ spanned by

$$\{m_{\mathfrak{s}\mathfrak{t}} \,|\, \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\mu}) \text{ for some } \boldsymbol{\mu} \in Par_{r,n} \text{ and } \boldsymbol{\mu} \rhd \boldsymbol{\lambda}\}.$$

Then one checks using Lemma 10 that $\mathcal{Y}^{\boldsymbol{\lambda}}_{r,n}$ is an ideal of $\mathcal{Y}_{r,n}(q)$. Using Lemma 10 once again, we get for $h \in \mathcal{Y}_{r,n}(q)$ the formula

$$m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} h = \sum_{\mathfrak{v}} r_{\mathfrak{v}} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \mod \widehat{\mathcal{Y}^{\boldsymbol{\lambda}}_{r,n}}$$

where $r_{\mathfrak{v}} \in R$. This is so because $\mathbf{t}^{\boldsymbol{\lambda}}$ is a maximal element of $\mathrm{Std}(\boldsymbol{\lambda})$. Multiplying this equation on the left with $g^*_{d(\mathfrak{s})}$ we get the formula

$$m_{\mathfrak{s}\mathfrak{t}}h = \sum_{\mathfrak{v}} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \mod \widehat{\mathcal{Y}^{\boldsymbol{\lambda}}_{r,n}}$$

and this is the multiplicative property that is required for cellularity. □

As already explained in [**16**], the existence of a cellular basis in an algebra A has strong consequences for the modular representation theory of A. Here we give two application of our cellular basis $\mathcal{B}_{r,n}$. The first one goes in a somewhat different direction, obtaining from it Lusztig's idempotent presentation of $\mathcal{Y}_{r,n}(q)$, used in [**30**] and [**31**].

PROPOSITION 2.2. *The Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}(q)$ is isomorphic to the associative R-algebra generated by the elements $\{g_i \mid i = 1,\dots,n-1\}$ and $\{f_{\mathfrak{s}} \mid \mathfrak{s} \in \mathrm{Std}^1_{n,r}\}$ subject to the following relations:*

$$g_i g_j = g_j g_i \qquad \text{for } |i-j|>1 \tag{3.27}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad \text{for all } i = 1,\dots,n-2 \tag{3.28}$$

$$f_{\mathfrak{s}} g_i = g_i f_{\mathfrak{s}s_i} \qquad \text{for all } \mathfrak{s}, i \tag{3.29}$$

$$g_i^2 = 1 + (q-q^{-1}) \sum_{\mathfrak{s}\in \mathrm{Std}^1_{n,r}} \delta_{i,i+1}(\mathfrak{s}) f_{\mathfrak{s}} g_i \qquad \text{for all } i \tag{3.30}$$

$$\sum_{\mathfrak{s}\in \mathrm{Std}^1_{n,r}} f_{\mathfrak{s}} = 1 \qquad \text{for all } \mathfrak{s} \tag{3.31}$$

$$f_{\mathfrak{s}} f_{\mathfrak{s}'} = \delta_{\mathfrak{s},\mathfrak{s}'} f_{\mathfrak{s}} \qquad \text{for all } \mathfrak{s},\mathfrak{s}' \in \mathrm{Std}^1_{n,r} \tag{3.32}$$

where $\delta_{\mathfrak{s},\mathfrak{s}'}$ is the Kronecker delta function on $\mathrm{Std}^1_{n,r}$ and where we set $\delta_{i,i+1}(\mathfrak{s}) := 1$ if i and $i+1$ belong to the same component (column) of $\mathfrak{s}$, otherwise $\delta_{i,i+1}(\mathfrak{s}) := 0$. Moreover, we define $f_{\mathfrak{s}s_i} := f_{\mathfrak{s}}$ if $\delta_{i,i+1}(\mathfrak{s}) = 0$.

PROOF. Let $\mathcal{Y}'_{r,n}$ be the R-algebra defined by the presentation of the Lemma. Then there is an R-algebra homomorphism $\varphi : \mathcal{Y}'_{r,n} \to \mathcal{Y}_{r,n}(q)$, given by $\varphi(g_i) := g_i$ and $\varphi(f_{\mathfrak{s}}) := m_{\mathfrak{s}\mathfrak{s}}$. Indeed, the $m_{\mathfrak{s}\mathfrak{s}}$'s are orthogonal idempotents and have sum 1 as we see from (3.23) and (3.25) respectively. Moreover, using (3.20), (3.23) and (3.25) we get that the relations (3.29), (3.30), (3.31) and (3.32) hold with $m_{\mathfrak{s}\mathfrak{s}}$ replacing $f_{\mathfrak{s}}$, and finally the first two relations hold trivially.

On the other hand, using (3.26) we get that φ is a surjection and since $\mathcal{Y}'_{r,n}$ is generated over R by the set $\{g_w f_{\mathfrak{s}} \mid w \in \mathfrak{S}_n, \mathfrak{s} \in \mathrm{Std}^1_{n,r}\}$ of cardinality $r^n n!$, we get that φ is also an injection. □

REMARK 6. *The relations given in the proposition are the relations, for type A, of the algebra H_n considered in 31.2 of* [**30**] *see also* [**34**]. *We would like to draw the attention to the sum appearing in the quadratic relation (3.30), making it look rather different than the quadratic relation of Yokonuma's or Juyumaya's presentation. In 31.2 of* [**30**], *it is mentioned that H_n is closely related to the convolution algebra associated with a Chevalley group and its unipotent* *radical and indeed in 35.3 of* [**31**]*, elements of this algebra are found that satisfy the relations of H_n. However, we could not find a Theorem in loc. cit., stating explicitly that H_n is isomorphic to $\mathcal{Y}_{r,n}(q)$. (On the other hand, in* [**22**] *Jacon and Poulain d'Andecy have recently given a simple explanation of the isomorphism $H_n \cong \mathcal{Y}_{r,n}(q)$).*

3.1. $\mathcal{Y}_{r,n}$ is a direct sum of matrix algebras. The second application of our cellular basis is to give an explicit isomorphism between the algebra $\mathcal{Y}_{r,n}(q)$ and the direct sum of matrix algebras $\mathrm{Mat}_{p_\mu}(\mathcal{H}_\mu(q))$ mentioned in Subsection 2.1. This result was first obtained by Lusztig in [**30**] using the above presentation of $\mathcal{Y}_{r,n}(q)$. Later, Jacon and Poulain d'Andecy gave an explicit isomorphism using the Juyumaya's presentation of $\mathcal{Y}_{r,n}(q)$ and certains idempotents indexed by the set of irreducible characters of $\mathcal{T}_n$. Unlike Jacon and Poulain d'Andecy's isomorphism, our isomorphism can be established over any specialization of $\mathcal{Y}_{r,n}(q)$ and it also preserves the cellular structure of these two algebras. In order to show this isomorphism, we need to introduce some notation.

Let $A = (I_1, \ldots, I_r)$ be an r-tuple of subsets of $\mathbf{n}$. We say that A is an ordered r-set partition of $\mathbf{n}$ if and only if $I_i \cap I_k = \emptyset$ when $i \neq k$ and $\coprod_{j=1}^r I_j = \mathbf{n}$. Let us denote by $\mathcal{SP}^{\mathrm{ord}}(n,r)$ the set of ordered r-set partitions of $\mathbf{n}$. Note that each $A \in \mathcal{SP}^{\mathrm{ord}}(n,r)$ has associated an unique set partition of $\mathbf{n}$ by considering only the non-empty components of A. We denote by $\{A\}$ the set partition associated to $A \in \mathcal{SP}^{\mathrm{ord}}(n,r)$. For example, if $A = (\{1,3,6\}, \emptyset, \{2,5\}, \emptyset, \emptyset, \{4\})$, then $\{A\} = \{\{1,3,6\}, \{2,5\}, \{4\}\}$

Now, for each $A = (I_1, \ldots, I_r) \in \mathcal{SP}^{\mathrm{ord}}(n,r)$ we can define

$$\mathbb{U}_A := E_{\{A\}} \prod_{d=1}^{r} u_{i_d, j_d} \tag{3.33}$$

where i_d is any element of the component I_{j_d}. Finally, to each r-multicomposition, $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ we can associate an ordered r-set partition $(I_1, \ldots, I_r)$ satisfying $|I_j| = |\lambda^{(j)}|$ for all $j = 1, \ldots, r$ where each non-empty component is defined as in (3.2). We denote this ordered r-set partition by $A^{ord}_{\boldsymbol{\lambda}}$.

As an immediate consequence of the definitions and Lemma 1 we get the following lemma

LEMMA 13. *The following hold*

(1) *For all $A \in \mathcal{SP}^{\mathrm{ord}}(n,r)$ and $1 \leq i \leq n-1$ we have $\mathbb{U}_A g_i = g_i \mathbb{U}_{A s_i}$, where $\mathfrak{S}_n$ acts on A by permutation on its numbers.*
(2) *For all $A \in \mathcal{SP}^{\mathrm{ord}}(n,r)$ we have that $\mathbb{U}_A = m_{\mathfrak{s}\mathfrak{s}}$ where $\mathfrak{s}$ is the unique multitableau in $\mathrm{Std}^1_{n,r}$ such that $A = A^{ord}_{\boldsymbol{\lambda}} d(\mathfrak{s})$.*
(3) *The set $\{\mathbb{U}_A \mid A \in \mathcal{SP}^{\mathrm{ord}}(n,r)\}$ is a set of orthogonal idempotents elements of $\mathcal{Y}_{r,n}$.*

PROOF. The statement (1) follows from relation (1.3) and Lemma 1. Suppose that $\mathfrak{s} \in \mathrm{Std}^1_{n,r}$ is a multitableau of shape $\boldsymbol{\lambda}$. Directly from the definitions we have that $x_{\boldsymbol{\lambda}} = 1$ and $\mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} = E_{\boldsymbol{\lambda}} U_{\boldsymbol{\lambda}}$, then

$$m_{\mathfrak{s}\mathfrak{s}} = g^*_{d(\mathfrak{s})} E_{\boldsymbol{\lambda}} U_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} = g^*_{d(\mathfrak{s})} \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} g_{d(\mathfrak{s})} = g^*_{d(\mathfrak{s})} g_{d(\mathfrak{s})} \mathbb{U}_A = \mathbb{U}_A$$

The last equality follows by an inductive argument over the length of $d(\mathfrak{s})$ using the multiplication rule of Lemma 6. The statement (3) follows immediate from the definitions. $\square$

We say that $A = (I_1, \dots, I_r)$ has type $\alpha \in \mathrm{comp}_r(n)$ if $(|I_1|, \dots, |I_r|) = \alpha$. For each $\alpha \in \mathrm{comp}_r(n)$, we denote by $\mathcal{SP}^{\mathrm{ord}}(n, \alpha)$ the set of ordered r-set partitions of n of type α. In particular, we have that

$$\mathcal{SP}^{\mathrm{ord}}(n, r) = \coprod_{\alpha \in \mathrm{comp}_r(n)} \mathcal{SP}^{\mathrm{ord}}(n, \alpha) \tag{3.34}$$

Taking all these definitions into account we can rewrite the elements, $m^{\boldsymbol{\lambda}}_{\mathfrak{s}\mathfrak{t}}$, of the cellular basis $\mathcal{B}_{r,n}$ as follows

$$m^{\boldsymbol{\lambda}}_{\mathfrak{s}\mathfrak{t}} = g^*_{d(\mathfrak{s})} \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})}$$

DEFINITION 2.6. *For all $\alpha = (n_1, \dots, n_r) \in \mathrm{comp}_r(n)$ we define*

$$\mathbb{U}_\alpha := \sum_{A \in \mathcal{SP}^{\mathrm{ord}}(n, \alpha)} \mathbb{U}_A$$

From Lemma 13 and (3.25) we have that $\{\mathbb{U}_\alpha \mid \alpha \in \mathrm{comp}_r(n)\}$ is a complete set of central orthogonal idempotents of $\mathcal{Y}_{r,n}$. As an immediate consequence we have that $\mathcal{Y}_{r,n}$ can be decomposed as a direct sum of two-sided ideals

$$\mathcal{Y}_{r,n} = \bigoplus_{\alpha \in \mathrm{comp}_r(n)} \mathbb{U}_\alpha \mathcal{Y}_{r,n} \tag{3.35}$$

Moreover, each R-subalgebra $\mathcal{Y}^\alpha_{r,n} := \mathbb{U}_\alpha \mathcal{Y}_{r,n}$ is a cellular R-algebra with cellular basis

$$\mathcal{B}^\alpha_{r,n} = \left\{ m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}),\ \boldsymbol{\lambda} \text{ is a multipartition of } n \text{ of type } \alpha \right\}$$

In particular, $\mathcal{Y}^\alpha_{r,n}$ is a free R-algebra of dimension $p_\alpha n!$, where p_α is the multinomial coefficient associated with α, defined as in (2.17).

From now on, we focus our study on the subalgebras $\mathcal{Y}^\alpha_{r,n}$ of $\mathcal{Y}_{r,n}$. For this, it is convenient to introduce some notation. For each $\alpha \in \mathrm{comp}_r(n)$, we denote by $Par_{r,n}(\alpha)$ the set of r-multipartition of n having type α.

LEMMA 14. *Let $\alpha \in \mathrm{comp}_r(n)$ and suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in Par_{r,n}(\alpha)$. If $\mathfrak{s}$ is a row-standar $\boldsymbol{\lambda}$-multitableau and $\mathfrak{t}$ is a row-standar $\boldsymbol{\mu}$-multitableau, then*

$$m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\mu}}} = \begin{cases} \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} x_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}_0} x_{\mathfrak{t}_0\mathfrak{t}^{\boldsymbol{\mu}}} & \text{if } w_{\mathfrak{s}} = w_{\mathfrak{t}} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathfrak{s}_0$ and $\mathfrak{t}_0$ are the multitableaux of initial kind associated with $\mathfrak{s}$ and $\mathfrak{t}$, respectively.

PROOF. We know that there exists $\boldsymbol{\mu}$-multitableaux $\mathfrak{s}_0$ and $\mathfrak{t}_0$ of initial kind together with distinguished right coset representative $w_{\mathfrak{s}}$ and $w_{\mathfrak{t}}$ for $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$ such that $\mathfrak{s} = \mathfrak{s}_0 w_{\mathfrak{s}}$ and $\mathfrak{t} = \mathfrak{t}_0 w_{\mathfrak{t}}$. By expanding the left-hand side of the statement we have

$$m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\mu}}} = \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{s}_0)} g_{w_{\mathfrak{s}}} g^*_{w_{\mathfrak{t}}} g^*_{d(\mathfrak{t}_0)} \mathbb{U}_{A^{ord}_{\boldsymbol{\mu}}} x_{\boldsymbol{\mu}} = x_{\boldsymbol{\lambda}} g_{d(\mathfrak{s}_0)} g_{w_{\mathfrak{s}}} \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}{}^{w_{\mathfrak{s}}}} \mathbb{U}_{A^{ord}_{\boldsymbol{\mu}}{}^{w_{\mathfrak{t}}}} g^*_{w_{\mathfrak{t}}} g^*_{d(\mathfrak{t}_0)} x_{\boldsymbol{\mu}} \tag{3.36}$$

Since the $\mathbb{U}_A$ are orthogonal idempotents, the right-hand side of (3.36) is nonzero if and only if $A^{ord}_{\boldsymbol{\lambda}} w_{\mathfrak{s}} = A^{ord}_{\boldsymbol{\mu}} w_{\mathfrak{t}}$. That is, $w_{\mathfrak{s}}$ and $w_{\mathfrak{t}}$ belong to the same orbit of $\mathfrak{S}_\alpha \backslash \mathfrak{S}_n$. But, by construction, both $w_{\mathfrak{s}}$ and $w_{\mathfrak{t}}$ are coset representative of minimal length, which implies that $w_{\mathfrak{s}} = w_{\mathfrak{t}}$. Now, if $w_{\mathfrak{s}} = w_{\mathfrak{t}}$ we have that

$$\mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} g_{w_{\mathfrak{s}}} g^*_{w_{\mathfrak{t}}} = \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} g_{w_{\mathfrak{s}}} g_{w_{\mathfrak{s}}^{-1}} = \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}}.$$

Finally, using the above and reordering in the last equality of (3.36), we conclude that

$$m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\mu}}} = \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} x_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}_0} x_{\mathfrak{t}_0 \mathfrak{t}^{\boldsymbol{\mu}}}$$

□

Recall that for each $\mu = (\mu_1, \dots, \mu_r) \in \text{comp}_r(n)$ we have an Young-Hecke algebra $\mathcal{H}_\alpha$, which is spanned by the elements h_i's such that $s_i \in \Sigma \cap \mathfrak{S}_\alpha$. Then $\mathcal{H}_\alpha$ is naturally isomorphic to the tensorial product of Hecke algebras H_{α_i}, with $i = 1, \dots, r$. We note that $\mathcal{H}_\alpha$ can also be seen as the free R-module with basis $\{h_w \mid w \in \mathfrak{S}_\alpha\}$. Moreover, from the general theory of cellular algebra we have that each $\mathcal{H}_\alpha$ is a cellular algebra with cellular basis

$$\mathcal{C}_\alpha = \{x^{\lambda_1}_{\mathfrak{s}_1 \mathfrak{t}_1} x^{\lambda_2}_{\mathfrak{s}_2 \mathfrak{t}_2} \cdots x^{\lambda_r}_{\mathfrak{s}_r \mathfrak{t}_r} \mid \mathfrak{s}_i, \mathfrak{t}_i \in \text{Std}(\lambda_i), \lambda_i \in \mathcal{P}ar_{\alpha_i}, i = 1, \dots, r\} \tag{3.37}$$

where, for each $\mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda)$ and $\lambda \in \mathcal{P}_n$, we denote by $x_{\mathfrak{s}\mathfrak{t}}$ the Murphy elements of $\mathcal{H}_n$.

We know that each element of the cellular basis of $\mathcal{Y}^\alpha_{r,n}$ can be written in the following form

$$m^{\boldsymbol{\lambda}}_{\mathfrak{s}\mathfrak{t}} = g^*_{d(\mathfrak{s})} \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})} = g^*_{w_{\mathfrak{s}}} \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} (g^*_{d(\mathfrak{s}_0)} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{t}_0)}) g_{w_{\mathfrak{t}}} = g^*_{w_{\mathfrak{s}}} \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} x_{\mathfrak{s}_0 \mathfrak{t}_0} g_{w_{\mathfrak{t}}} \tag{3.38}$$

where $\mathfrak{t}_0$ y $\mathfrak{s}_0$ are multitableaux of initial kind and $w_{\mathfrak{s}}$, $w_{\mathfrak{t}}$ are distinguished right coset representative for $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$.

Finally, we note that the cardinal of the set $\{w_{\mathfrak{s}} \mid s \in \text{Std}(\boldsymbol{\lambda}), \|\boldsymbol{\lambda}\| = \alpha\}$ is equal to the multinomial coefficient p_α. We can introduce an arbitrary total orden on $\{w_{\mathfrak{s}}\}$ and denote by $M_{\mathfrak{s}\mathfrak{t}}$ the elementary matrix of $\text{Mat}_{p(\alpha)}(\mathcal{H}_\alpha(q))$ which is equal to 1 at the intersection of the row and column indexed by $w_{\mathfrak{s}}$ and $w_{\mathfrak{t}}$, and 0 otherwise. Then, the decomposition (3.38) and Lemma 27 implies the following result.

THEOREM 2.7. *Let $\alpha \in \text{comp}_r(n)$. We have the following isomorphism of R-algebras*

$$\begin{array}{cccc} \Phi_\alpha : & \mathcal{Y}^\alpha_{r,n} & \longrightarrow & \text{Mat}_{p_\alpha}(\mathcal{H}_\alpha) \\ & m^{\boldsymbol{\lambda}}_{\mathfrak{s}\mathfrak{t}} = g^*_{w_{\mathfrak{s}}} \mathbb{U}_{A^{ord}_{\boldsymbol{\lambda}}} x_{\mathfrak{s}_0 \mathfrak{t}_0} g_{w_{\mathfrak{t}}} & \to & x_{\mathfrak{s}_0 \mathfrak{t}_0} M_{\mathfrak{s},\mathfrak{t}} \end{array}$$

PROOF. Since Φ_α maps a (cellular) basis of $\mathcal{Y}^\alpha_{r,n}$ to a (cellular) basis of $\text{Mat}_{p_\alpha}(\mathcal{H}_\alpha)$, it is clear that Φ_α is an R-linear isomorphism. To complete the proof of the theorem, we need only show that Φ_α preserves multiplication. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be multipartitions of α-type. From Lemma 27 it follows that for each pair of standard $\boldsymbol{\lambda}$-multitableaux $\mathfrak{s}$ and $\mathfrak{t}$, and each pair of standard $\boldsymbol{\mu}$-multitableaux $\mathfrak{a}$ and $\mathfrak{b}$, we have

$$m^{\boldsymbol{\lambda}}_{\mathfrak{s}\mathfrak{t}} m^{\boldsymbol{\mu}}_{\mathfrak{a}\mathfrak{b}} = \begin{cases} g^*_{w_{\mathfrak{s}}} \mathbb{E}_{A^{\boldsymbol{\lambda}}} x_{\mathfrak{s}_0 \mathfrak{t}_0} x_{\mathfrak{a}_0 \mathfrak{b}_0} g_{w_{\mathfrak{b}}} & \text{if } w_{\mathfrak{t}} = w_{\mathfrak{a}} \\ 0 & \text{if } w_{\mathfrak{t}} \neq w_{\mathfrak{a}} \end{cases}$$

On the other hand, it is well known that $M_{\mathfrak{s},\mathfrak{t}}M_{\mathfrak{a},\mathfrak{b}}=\delta_{\mathfrak{t}\mathfrak{a}}M_{\mathfrak{s},\mathfrak{b}}$ where $\delta_{\mathfrak{t}\mathfrak{a}}$ is the Dirac's delta function. Therefore, it is immediate that

$$\Phi_\alpha(m^{\boldsymbol{\lambda}}_{\mathfrak{s}\mathfrak{t}})\Phi_\alpha(y^{\boldsymbol{\mu}}_{\mathfrak{a}\mathfrak{b}})=\begin{cases} x_{\mathfrak{s}_0\mathfrak{t}_0}x_{\mathfrak{a}_0\mathfrak{b}_0}M_{\mathfrak{s},\mathfrak{b}} & \text{if } w_{\mathfrak{t}}=w_{\mathfrak{a}}\\ 0 & \text{if } w_{\mathfrak{t}}\neq w_{\mathfrak{a}}\end{cases}$$

Finally, we note that the equality $\Phi_\alpha(m^{\boldsymbol{\lambda}}_{\mathfrak{s}\mathfrak{t}}m^{\boldsymbol{\mu}}_{\mathfrak{a}\mathfrak{b}})=\Phi_\alpha(m^{\boldsymbol{\lambda}}_{\mathfrak{s}\mathfrak{t}})\Phi_\alpha(m^{\boldsymbol{\mu}}_{\mathfrak{a}\mathfrak{b}})$ is obtained by expanding the product $x_{\mathfrak{s}_0\mathfrak{t}_0}x_{\mathfrak{a}_0\mathfrak{b}_0}$ in $\Phi_\alpha(m^{\boldsymbol{\lambda}}_{\mathfrak{s}\mathfrak{t}}y^{\boldsymbol{\mu}}_{\mathfrak{a}\mathfrak{b}})$ and then by applying directly the R-linearity of Φ_α. □

The following result is immediate from the above theorem and decomposition (3.35).

COROLLARY 2.2. *The linear map*

$$\Phi:=\bigoplus_{\alpha\in\mathrm{comp}_r(n)}\Phi_\alpha:\mathcal{Y}_{r,n}\to\bigoplus_{\alpha\in\mathrm{comp}_r(n)}\mathrm{Mat}_{p_\alpha}(\mathcal{H}_\alpha)$$

is an isomorphism of R-algebras.

4. Jucys-Murphy elements

In this section we show that the Jucys-Murphy elements J_i for $\mathcal{Y}_{r,n}(q)$, introduced by Chlouveraki and Poulain d'Andecy in [**8**], are JM-elements in the abstract sense defined by Mathas, see [**36**]. This is with respect to the cellular basis for $\mathcal{Y}_{r,n}(q)$ obtained in the previous section.

We first consider the elements J'_k of $\mathcal{Y}_{r,n}(q)$ given by $J'_1=0$ and for $k\geq 1$

$$J'_{k+1}=q^{-1}(e_kg_{(k,k+1)}+e_{k-1,k+1}g_{(k-1,k+1)}+\cdots+e_{1,k+1}g_{(1,k+1)}) \tag{4.1}$$

where $g_{(i,k+1)}$ is g_w for $w=(i,k+1)$. These elements are generalizations of the Jucys-Murphy elements for the Iwahori-Hecke algebra $\mathcal{H}_n(q)$, in the sense that we have $E_{\mathbf{n}}J'_k=E_{\mathbf{n}}L_k$, where L_k are the Jucys-Murphy elements for $\mathcal{H}_n(q)$ defined in [**35**].

The elements J_i of $\mathcal{Y}_{r,n}(q)$ that we shall refer to as Jucys-Murphy elements were introduced by Chlouveraki and Poulain d'Andecy in [**8**] via the recursion

$$J_1=1 \quad\text{and}\quad J_{i+1}=g_iJ_ig_i \quad \text{for } i=1,\dots,n-1. \tag{4.2}$$

The relation between J_i and J'_i is given by

$$J_i=1+(q^2-1)J'_i. \tag{4.3}$$

In fact, in [**8**] the elements $\{J_1,\dots,J_n\}$, as well as the elements $\{t_1,\dots,t_n\}$, are called Jucys-Murphy elements for the Yokonuma-Hecke algebra.

The following definition appears for the first time in [**36**]. It formalizes the concept of Jucys-Murphy elements.

DEFINITION 2.7. *Suppose that the $\mathcal{R}$-algebra A is cellular with antiautomorphism $*$ and cellular basis $\mathcal{C} = \{a_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda, \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$. Suppose moreover that each set $T(\lambda)$ is endowed with a poset structure with order relation $\rhd_\lambda$. Then we say that a commuting set $\mathcal{L} = \{L_1, \dots, L_M\} \subseteq A$ is a family of JM-elements for A, with respect to the basis $\mathcal{C}$, if it satisfies that $L_i^* = L_i$ for all i and if there exists a set of scalars $\{c_\mathfrak{t}(i) \mid \mathfrak{t} \in T(\lambda),\ 1 \le i \le M\}$, called the contents of λ, such that for all $\lambda \in \Lambda$ and $\mathfrak{t} \in T(\lambda)$ we have that*

$$a_{\mathfrak{s}\mathfrak{t}} L_i = c_\mathfrak{t}(i) a_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{v} \in T(\lambda) \\ \mathfrak{v} \rhd_\lambda \mathfrak{t}}} r_{\mathfrak{s}\mathfrak{v}} a_{\mathfrak{s}\mathfrak{v}} \mod A^\lambda \tag{4.4}$$

for some $r_{\mathfrak{s}\mathfrak{v}} \in \mathcal{R}$.

Our goal is to prove that the set

$$\mathcal{L}_{\mathcal{Y}_{r,n}} := \{L_1, \dots, L_{2n} \mid L_k = J_k,\ L_{n+k} = t_k,\ 1 \le k \le n\} \tag{4.5}$$

is a family of JM-elements for $\mathcal{Y}_{r,n}(q)$ in the above sense. Let us start out by stating the following Lemma.

LEMMA 15. *Let i and k be integers such that $1 \le i < n$ and $1 \le k \le n$. Then*

(1) *g_i and J_k commute if $i \neq k-1, k$.*
(2) *$\mathcal{L}_{\mathcal{Y}_{r,n}}$ is a set of commuting elements.*
(3) *g_i commutes with $J_i J_{i+1}$ and $J_i + J_{i+1}$.*
(4) *$g_i J_i = J_{i+1} g_i + (q^{-1} - q) e_i J_{i+1}$ and $g_i J_{i+1} = J_i g_i + (q - q^{-1}) e_i J_{i+1}$.*

PROOF. For the proof of (1) and (2), see [**8**, Corollaries 1 and 2]. We then prove (3) using (1) and (2) and induction on i. For $i = 1$ the two statements are trivial. For $i > 1$ we have that

$$\begin{aligned} g_i J_i J_{i+1} &= g_i (g_{i-1} J_{i-1} g_{i-1})(g_i g_{i-1} J_{i-1} g_{i-1} g_i) = g_i g_{i-1} J_{i-1} g_i g_{i-1} g_i J_{i-1} g_{i-1} g_i \\ &= g_i g_{i-1} g_i J_{i-1} g_{i-1} g_i J_{i-1} g_{i-1} g_i = g_{i-1} (g_i g_{i-1} J_{i-1} g_{i-1} g_i) J_{i-1} g_{i-1} g_i \\ &= g_{i-1} J_{i+1} J_{i-1} g_{i-1} g_i = (g_{i-1} J_{i-1} g_{i-1}) J_{i+1} g_i = J_i J_{i+1} g_i \end{aligned}$$

and

$$\begin{aligned} &g_i (J_i + J_{i+1}) = g_i J_i + g_i^2 J_i g_i = g_i J_i + (1 + (q - q^{-1}) e_i g_i) J_i g_i \\ &= J_i g_i + g_i J_i (1 + (q - q^{-1}) e_i g_i) = J_i g_i + g_i J_i g_i^2 = (J_i + J_{i+1}) g_i. \end{aligned}$$

Finally, the equalities of (4) are also shown by using (2) and direct computations, as we show next

$$g_i J_i g_i^{-1} = g_i J_i (g_i + (q^{-1} - q) e_i) = J_{i+1} + (q^{-1} - q) g_i J_i e_i = J_{i+1} + (q^{-1} - q) e_i g_i J_i$$

and

$$g_i J_{i+1} g_i^{-1} = g_i^2 J_i = (1 + (q - q^{-1}) e_i g_i) J_i = J_i + (q - q^{-1}) e_i g_i J_i$$

Then, both equalities are obtained by multiplying on the right with g_i in the above equalities. □

Let $\mathcal{K}$ be an R-algebra as above, such that $q \in \mathcal{K}^\times$. Let $\mathbf{t}$ be a $\boldsymbol{\lambda}$-multitableau and suppose that the node of $\mathbf{t}$ labelled by (x, y, k) is filled in with j. Then we define the *quantum content* of j as the element $c_{\mathfrak{t}}(j) := q^{2(y-x)} \in \mathcal{K}$. We furthermore define $\mathrm{res}_{\mathfrak{t}}(j) := y-x$ and then have the formula $c_{\mathfrak{t}}(j) = q^{2\mathrm{res}_{\mathfrak{t}}(j)}$. When $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$, we write $c_{\boldsymbol{\lambda}}(j)$ for $c_{\mathfrak{t}}(j)$.

The next Proposition is the main result of this section.

PROPOSITION 2.3. *$(\mathcal{Y}_{r,n}(q), \mathcal{B}_{r,n})$ is a cellular algebra with family of JM-elements $\mathcal{L}_{\mathcal{Y}_{r,n}}$ and contents given by*

$$d_{\mathfrak{t}}(k) := \begin{cases} c_{\mathfrak{t}}(k) & \text{if } k = 1, \dots, n \\ \xi^{p_{\mathfrak{t}}(k)} & \text{if } k = n+1, \dots, 2n. \end{cases}$$

PROOF. We have already proved that $\mathcal{B}_{r,n}$ is a cellular basis for $\mathcal{Y}_{r,n}(q)$, so we only need to prove that the elements of $\mathcal{L}_{\mathcal{Y}_{r,n}}$ verify the conditions of Definition 2.7.

For the order relation $\rhd_{\boldsymbol{\lambda}}$ on $\mathrm{Std}(\boldsymbol{\lambda})$ we shall use the dominance order $\rhd$ on multitableaux that was introduced above. By Lemma 12 the JM-condition (4.4) holds for $k = n+1, \dots, 2n$ and so we only need to check the cases $k = 1, \dots, n$.

Let us first consider the case when $\mathfrak{t}$ is a standard $\boldsymbol{\lambda}$-multitableau of the initial kind. Suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$, $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$ and $\alpha = \|\boldsymbol{\lambda}\|$, with corresponding Young subgroup $\mathfrak{S}_{\alpha} = \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_r}$ and suppose that k belongs to $\mathfrak{t}^{(l)}$. Since $\mathfrak{t}$ is of the initial kind we have from (3.17) a corresponding decomposition

$$m_{\boldsymbol{\lambda}\mathfrak{t}} = U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} x_{\lambda^{(1)}\mathfrak{t}^{(1)}} x_{\lambda^{(2)}\mathfrak{t}^{(2)}} \cdots x_{\lambda^{(r)}\mathfrak{t}^{(r)}} \tag{4.6}$$

where, as before, $\boldsymbol{\lambda}$ and $\lambda^{(i)}$ as indices refer to $\mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}^{\lambda^{(i)}}$. Hence, by (1) of Lemma 15 we get that

$$\begin{gathered} m_{\boldsymbol{\lambda}\mathfrak{t}} J_k = U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} x_{\lambda^{(1)}\mathfrak{t}^{(1)}} \cdots x_{\lambda^{(l)}\mathfrak{t}^{(l)}} J_k x_{\lambda^{(l+1)}\mathfrak{t}^{(l+1)}} \cdots x_{\lambda^{(r)}\mathfrak{t}^{(r)}} = \\ x_{\lambda^{(1)}\mathfrak{t}^{(1)}} \cdots x_{\lambda^{(l)}\mathfrak{t}^{(l)}} U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} (1 + (q^2-1) J'_k) x_{\lambda^{(l+1)}\mathfrak{t}^{(l+1)}} \cdots x_{\lambda^{(r)}\mathfrak{t}^{(r)}} \end{gathered} \tag{4.7}$$

where we used Lemma 1 to commute $U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}}$ past $x_{\lambda^{(1)}\mathfrak{t}^{(1)}} \cdots x_{\lambda^{(l)}\mathfrak{t}^{(l)}}$. On the other hand, by Lemma 8 together with the definition of J'_i we have that

$$U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} x_{\lambda^{(l)}\mathfrak{t}^{(l)}} (1 + (q^2-1) J'_k) = U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} x_{\lambda^{(l)}\mathfrak{t}^{(l)}} (1 + (q^2-1) L^l_k) \tag{4.8}$$

where $L^l_k = q^{-1}(g_{(k,k+1)} + g_{(k-1,k+1)} + \cdots + g_{(m,k+1)})$ is the k'th Jucys-Murphy element as in [**35**] for the Iwahori-Hecke algebra corresponding to $\mathfrak{S}_{\alpha_l}$, permuting the numbers $\{m, m+1, \dots, m+\alpha_l-1\}$. Thus under the isomorphism $\mathcal{H}_{\alpha_1}(q) \otimes \cdots \otimes \mathcal{H}_{\alpha_r}(q) \cong \mathcal{Y}_{\alpha}(q)$ of Lemma 7 we have that the l'the factor of $m_{\boldsymbol{\lambda}\mathfrak{t}} J_k \in \mathcal{Y}_{\alpha}(q)$ is $x_{\lambda^{(l)}\mathfrak{t}^{(l)}}(1 + (q^2-1) L^l_k) \in \mathcal{H}_{\alpha_l}(q)$ and so we may further manipulate that element inside $\mathcal{H}_{\alpha_l}(q)$.

Now applying [**35**, Theorem 3.32] we get that $x_{\lambda^{(l)}\mathfrak{t}^{(l)}}(1 + (q^2-1) L^l_k)$ is equal to

$$\begin{aligned} & x_{\lambda^{(l)}\mathfrak{t}^{(l)}} + (q^2-1)[\mathrm{res}_{\mathfrak{t}^{(l)}}(k)]_q x_{\lambda^{(l)}\mathfrak{t}^{(l)}} + \sum_{\substack{\mathfrak{v} \in \mathrm{Std}(\lambda^{(l)}) \\ \mathfrak{v} \rhd \mathfrak{t}^{(l)}}} a_{\mathfrak{v}} x_{\lambda^{(l)}\mathfrak{v}} + \sum_{\substack{\mathfrak{a}_1, \mathfrak{b}_1 \in \mathrm{Std}(\mu^{(l)}) \\ \mu^{(l)} \rhd \lambda^{(l)}}} r_{\mathfrak{a}_1\mathfrak{b}_1} x_{\mathfrak{a}_1\mathfrak{b}_1} \\ & = q^{2(\mathrm{res}_{\mathfrak{t}^{(l)}}(k))} x_{\lambda^{(l)}\mathfrak{t}^{(l)}} + \sum_{\substack{\mathfrak{v} \in \mathrm{Std}(\lambda^{(l)}) \\ \mathfrak{v} \rhd \mathfrak{t}^{(l)}}} a_{\mathfrak{v}} x_{\lambda^{(l)}\mathfrak{v}} + \sum_{\substack{\mathfrak{a}_1, \mathfrak{b}_1 \in \mathrm{Std}(\mu^{(l)}) \\ \mu^{(l)} \rhd \lambda^{(l)}}} r_{\mathfrak{a}_1\mathfrak{b}_1} x_{\mathfrak{a}_1\mathfrak{b}_1} \end{aligned} \tag{4.9}$$

for some $r_{\mathfrak{a}_1\mathfrak{b}_1}, a_{\mathfrak{v}} \in R$ where the tableaux $\mathfrak{a}_1, \mathfrak{b}_1 \in \mathrm{Std}(\mu^{(l)})$ involve the numbers permuted by $\mathfrak{S}_{\alpha_l}$. For $\mathfrak{a}_1, \mathfrak{b}_1$ and $\mathfrak{v}$ appearing in the sum set $\mathfrak{a} := (\mathfrak{t}^{\lambda^{(1)}}, \dots, \mathfrak{a}_1, \dots, \mathfrak{t}^{\lambda^{(r)}})$, $\mathfrak{b} := (\mathfrak{t}^{(1)}, \dots, \mathfrak{b}_1, \dots, \mathfrak{t}^{(r)})$

and $\mathfrak{c} := (\mathfrak{t}^{\lambda^{(1)}}, \dots, \mathfrak{v}, \dots, \mathfrak{t}^{\lambda^{(r)}})$. Then $\mathfrak{c} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $\mathfrak{a}, \mathfrak{b} \in \mathrm{Std}(\boldsymbol{\mu})$ where $\boldsymbol{\mu} := (\lambda^{(1)}, \dots, \mu^{(l)}, \dots \lambda^{(r)})$. Moreover, by our definition of the dominance order we have $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$, $\mathfrak{c} \rhd \mathfrak{t}$ and so $m_{\mathfrak{a}\mathfrak{b}} \in \mathcal{Y}^{\boldsymbol{\lambda}}_{r,n}$. On the other hand, we have

$$U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} x_{\lambda^{(1)}\mathfrak{t}^{(1)}} \cdots x_{\mathfrak{a}_1 \mathfrak{b}_1} \cdots x_{\lambda^{(r)}\mathfrak{t}^{(r)}} = U_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}} g^*_{d(\mathfrak{a}_1)} x_{\lambda^{(1)}\mathfrak{t}^{(1)}} \cdots x_{\mu^{(l)}} \cdots x_{\lambda^{(r)}\mathfrak{t}^{(r)}} g_{d(\mathfrak{b}_1)} = m_{\mathfrak{a}\mathfrak{b}}$$

and similarly for $m_{\boldsymbol{\lambda}\mathfrak{t}}$ and $m_{\boldsymbol{\lambda}\mathfrak{c}}$. Using Lemma 7 in the other direction together with $\mathrm{res}_{\mathfrak{t}^{(l)}}(k) = \mathrm{res}_{\mathfrak{t}}(k)$ we then get

$$m_{\boldsymbol{\lambda}\mathfrak{t}} J_k \quad = \quad c_{\mathfrak{t}}(k) m_{\boldsymbol{\lambda}\mathfrak{t}} + \sum_{\substack{\mathfrak{c} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \mathfrak{c} \rhd \mathfrak{t}}} a_{\mathfrak{c}} m_{\boldsymbol{\lambda}\mathfrak{c}} \mod \mathcal{Y}^{\boldsymbol{\lambda}}_{r,n}$$

which shows the Proposition for $\mathfrak{t}$ of the initial kind.

For $\mathfrak{t}$ a general multitableau, there exists a multitableau $\mathfrak{t}_0$ of the initial kind together with a distinguished right coset representative $w_{\mathfrak{t}}$ of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}_0 w_{\mathfrak{t}}$. Let $w_{\mathfrak{t}} = s_{i_1} s_{i_2} \dots s_{i_k}$ be a reduced expression for $w_{\mathfrak{t}}$. Then we have that i_j and $i_j + 1$ are located in different blocks of $\mathfrak{t}_0 s_{i_1} \dots s_{i_{j-1}}$ for all $j \geq 1$ and that $\mathfrak{t}_0 s_{i_1} \dots s_{i_{j-1}} s_{i_j}$ is obtained from $\mathfrak{t}_1 s_{i_1} \dots s_{i_{j-1}}$ by interchanging i_j and $i_j + 1$. Using Lemma 8 and (4) of Lemma 15 we now get that

$$m_{\boldsymbol{\lambda}\mathfrak{t}} J_k = m_{\boldsymbol{\lambda}\mathfrak{t}_0} g_{w_{\mathfrak{t}}} J_k = m_{\boldsymbol{\lambda}\mathfrak{t}_0} J_{k w_{\mathfrak{t}}^{-1}} g_{w_{\mathfrak{t}}}.$$

Since $\mathfrak{t}_0$ is of the initial kind, we get

$$\begin{aligned} m_{\boldsymbol{\lambda}\mathfrak{t}} J_k \quad &= m_{\boldsymbol{\lambda}\mathfrak{t}_0} J_{k w_{\mathfrak{t}}^{-1}} g_{w_{\mathfrak{t}}} = \left(c_{\mathfrak{t}_0}(k w_{\mathfrak{t}}^{-1}) m_{\mathfrak{t}^{\lambda}\mathfrak{t}_0} + \sum_{\substack{\mathfrak{v}_0 \in \mathrm{Std}(\lambda) \\ \mathfrak{v}_0 \rhd \mathfrak{t}_0}} a_{\mathfrak{v}_0} m_{\boldsymbol{\lambda}\mathfrak{v}_0} \right) g_{w_{\mathfrak{t}}} \\ &= c_{\mathfrak{t}}(k) m_{\boldsymbol{\lambda}\mathfrak{t}} + \sum_{\substack{\mathfrak{v} \in \mathrm{Std}(\lambda) \\ \mathfrak{v} \rhd \mathfrak{t}}} a_{\mathfrak{v}} m_{\boldsymbol{\lambda}\mathfrak{v}} \end{aligned}$$

where we used that the occurring $\mathfrak{v}_0$ are all of the initial kind such that $m_{\mathfrak{v}} = m_{\mathfrak{v}_0} g_{w_{\mathfrak{t}}}$ with $\mathfrak{v} \rhd \mathfrak{t}_0$ and $a_{\mathfrak{v}} = a_{\mathfrak{v}_0}$. This finishes the proof of the Proposition. $\square$

In view of the Proposition, we can now apply the general theory developed in [**36**]. In particular, we recover the semisimplicity criterion of Chlouveraki and Poulain d'Andecy, [**8**], and can even generalize it to the case of ground fields of positive characteristic. We leave the details to the reader.

CHAPTER 3

Representation theory of the braids and ties algebra

1. Braids and ties algebra

As mentioned above, the specialized Yokonuma-Hecke algebra $\mathcal{Y}^{\mathcal{K}}_{r,n}(q)$ only exists if r is a unit in $\mathcal{K}$. The algebra of braids and ties $\mathcal{E}_n(q)$, introduced by Aicardi and Juyumaya, is an algebra related to $\mathcal{Y}_{r,n}(q)$ that exists for any ground ring. It has a diagram calculus consisting of braids that may be decorated with socalled ties, which explains its name, see [**1**]. Here we only give its definition in terms of generators and relations.

DEFINITION 3.1. *Let n be a positive integer. The algebra of braids and ties, $\mathcal{E}_n = \mathcal{E}_n(q)$, is the associative $S := \mathbb{Z}[q, q^{-1}]$-algebra generated by the elements $g_1, \dots, g_{n-1}, e_1, \dots, e_{n-1}$, subject to the following relations:*

$$
\begin{aligned}
g_i g_j &= g_j g_i && \text{for } |i-j| > 1 && (1.1)\\
g_i e_i &= e_i g_i && \text{for all } i && (1.2)\\
g_i g_j g_i &= g_j g_i g_j && \text{for } |i-j| = 1 && (1.3)\\
e_i g_j g_i &= g_j g_i e_j && \text{for } |i-j| = 1 && (1.4)\\
e_i e_j g_j &= e_i g_j e_i = g_j e_i e_j && \text{for } |i-j| = 1 && (1.5)\\
e_i e_j &= e_j e_i && \text{for all } i, j && (1.6)\\
g_i e_j &= e_j g_i && \text{for } |i-j| > 1 && (1.7)\\
e_i^2 &= e_i && \text{for all } i && (1.8)\\
g_i^2 &= 1 + (q - q^{-1}) e_i g_i && \text{for all } \ i. && (1.9)
\end{aligned}
$$

Once again, this differs slightly from the presentation normally used for $\mathcal{E}_n(q)$, for example in [**39**], where the variable u is used and the quadratic relation takes the form $\tilde{g}_i^2 = 1+(u-1)e_i(\tilde{g}_i+1)$. And once again, to change between the two presentations one uses $u = q^2$ and

$$g_i = \tilde{g}_i + (q^{-1} - 1) e_i \tilde{g}_i \tag{1.10}$$

For any commutative ring $\mathcal{K}$ containing the invertible element q, we define the specialized algebra $\mathcal{E}^{\mathcal{K}}_n(q)$ via $\mathcal{E}^{\mathcal{K}}_n(q) := \mathcal{E}_n(q) \otimes_S \mathcal{K}$ where $\mathcal{K}$ is made into an S-algebra by mapping $q \in S$ to $q \in \mathcal{K}$.

LEMMA 16. *Let $\mathcal{K}$ be a commutative ring containing invertible elements r, ξ, Δ as above. Then there is a homomorphim $\varphi = \varphi_{\mathcal{K}} : \mathcal{E}_n^{\mathcal{K}}(q) \longrightarrow \mathcal{Y}_{r,n}^{\mathcal{K}}(q)$ of $\mathcal{K}$-algebras induced by $\varphi(g_i) := g_i$ and $\varphi(e_i) := e_i$.*

PROOF. This is immediate from the relations. □

As a consequence of our tensor space module for $\mathcal{Y}_{r,n}(q)$, we have the following result.

THEOREM 3.1. *Suppose that $r \geq n$. Then the homomorphism $\varphi : \mathcal{E}_n^{\mathcal{K}}(q) \to \mathcal{Y}_{r,n}^{\mathcal{K}}(q)$ introduced in Lemma 16 is an embedding.*

In order to prove Theorem 3.1 we need to modify the proof of Corollary 4 of [**39**] to make it valid for general $\mathcal{K}$. For this we first prove the following Lemma.

LEMMA 17. *Let $\mathcal{K}$ be an R-algebra as above and let $A = (I_1, \dots, I_d) \in \mathcal{SP}_n$ be a set partition. Denote by V_A the $\mathcal{K}$-submodule of $V^{\otimes n}$ spanned by the vectors*

$$v_n^{j_n} \otimes \cdots \otimes v_k^{j_k} \otimes \cdots \otimes v_l^{j_l} \otimes \cdots \otimes v_1^{j_1} \qquad 0 \leq j_i \leq r-1$$

with decreasing lower indices and satisfying that $j_k = j_l$ exactly if k and l belong to the same block I_i of A. Let $E_A \in \mathcal{E}_n^{\mathcal{K}}(q)$ be the element defined the same way as $E_A \in \mathcal{Y}_{r,n}(q)$, that is via formula (1.16). Then for all $v \in V_A$ we have that $vE_A = v$ whereas $vE_B = 0$ for $B \in \mathcal{SP}_n$ satisfying $B \not\subseteq A$ with respect to the order $\subseteq$ introduced above.

PROOF. In order to prove the first statement it is enough to show that e_{kl} acts as the identity on the basis vectors of V_A whenever k and l belong to the same block of A. But this follows from the expression for e_{kl} given in (1.11) together with the definition (2.2) of the action of $\mathbf{G}_i$ on $V^{\otimes n}$ and Lemma 2. Just as in the proof of Theorem 2.2 we use that the action of $\mathbf{G}_i$ on $v \in V_A$ is just permutation of the i'th and $i+1$'st factors of v since the lower indices are decreasing.

In order to show the second statement, we first remark that the condition $B \not\subseteq A$ means that there exist k and l belonging to the same block of B, but to different blocks of A. In other words e_{kl} appears as a factor of the product defining E_B whereas for all basis vectors of V_A

$$v_n^{j_n} \otimes \cdots \otimes v_k^{j_k} \otimes \cdots \otimes v_l^{j_l} \otimes \cdots \otimes v_1^{j_1}$$

we have that $j_k \neq j_l$. Just as above, using that the action of $\mathbf{G}_i$ is given by place permutation when the lower indices are decreasing, we deduce from this that $V_A e_{kl} = 0$ and so finally that $V_A E_B = 0$, as claimed. □

PROOF OF THEOREM 3.1. It is enough to show that the composition $\rho^{\mathcal{K}} \circ \varphi$ is injective since we know from Theorem 2.2 that $\rho^{\mathcal{K}}$ is faithful. Now recall from Theorem 2 of [**39**] that the set $\{E_A g_w | A \in \mathcal{SP}_n, w \in \mathfrak{S}_n\}$ generates $\mathcal{E}_n(q)$ over $\mathbb{C}[q, q^{-1}]$ (it is even a basis). The proof of this does not involve any special properties of $\mathbb{C}$ and hence $\{E_A g_w | A \in \mathcal{SP}_n, w \in \mathfrak{S}_n\}$ also generates $\mathcal{E}_n^{\mathcal{K}}(q)$ over $\mathcal{K}$.

Let us now consider a nonzero element $\sum_{w,A} r_{w,A} E_A G_w$ in $\mathcal{E}_n^{\mathcal{K}}(q)$. Under $\rho^{\mathcal{K}} \circ \varphi$ it is mapped to $\sum_{w,A} r_{w,A} \mathbf{E}_A \mathbf{G}_w$ which we must show to be nonzero.

For this we choose $A_0 \in \mathcal{SP}_n$ satisfying $r_{w,A_0} \neq 0$ for some $w \in \mathfrak{S}_n$ and minimal with respect to this under our order $\subseteq$ on $\mathcal{SP}_n$. Let $v \in V_{A_0} \setminus \{0\}$ where V_{A_0} is defined as in the previous Lemma 17. Note that the condition $r \geq n$ ensures that $V_{A_0} \neq 0$, so such a v does exist. Then the Lemma gives us that

$$v\big(\sum_{w,A} r_{w,A} \mathbf{E}_A \mathbf{G}_w\big) = v\big(\sum_{w} r_{w,A_0} \mathbf{G}_w\big). \tag{1.11}$$

The lower indices of v are strictly decreasing and so each $\mathbf{G}_w$ acts on it by place permutation. It follows from this that (1.11) is nonzero, and the Theorem is proved. □

REMARK 7. *The above proof did not use the linear independence of* $\{E_A g_w | A \in \mathcal{SP}_n,$ $w \in \mathfrak{S}_n\}$ *over* $\mathcal{K}$. *In fact, it gives a new proof of Corollary 4 of* [**39**].

In the special case $\mathcal{K} = \mathbb{C}[q, q^{-1}]$ *and* $r = n$ *the Theorem is an immediate consequence of the faithfulness of the tensor product* $V^{\otimes n}$ *as an* $\mathcal{E}_n(q)$*-module, as proved in Corollary 4 of* [**39**]. *Indeed, let* $\rho_{\mathcal{E}_n}^{\mathbb{C}[q,q^{-1}]} : \mathcal{E}_n(q) \to \mathrm{End}(V^{\otimes n})$ *be the homomorphism associated with the* $\mathcal{E}_n(q)$*-module structure on* $V^{\otimes n}$, *introduced in* [**39**]. *Then the injectivity of* $\rho_{\mathcal{E}_n}^{\mathbb{C}[q,q^{-1}]}$ *together with the factorization* $\rho_{\mathcal{E}_n}^{\mathbb{C}[q,q^{-1}]} = \rho^{\mathbb{C}[q,q^{-1}]} \circ \varphi_{\mathbb{C}[q,q^{-1}]}$ *shows directly that* $\varphi_{\mathbb{C}[q,q^{-1}]}$ *is injective. One actually checks that the proof of Corollary 4 of* [**39**] *remains valid for* $\mathcal{K} = R$ *and* $r \geq n$, *but still this is not enough to prove injectivity of* $\varphi = \varphi_{\mathcal{K}}$ *for a general* $\mathcal{K}$ *since extension of scalars from* R *to* $\mathcal{K}$ *is not left exact. Note that the specialization argument of* [**39**] *fails for general* $\mathcal{K}$.

We shall often need the following relations in $\mathcal{E}_n(q)$, that have already appeared implicitly above

$$E_A g_w = g_w E_{Aw} \text{ and } E_A E_B = E_C \text{ for } w \in \mathfrak{S}_n, A, B \in \mathcal{SP}_n \tag{1.12}$$

where $C \in \mathcal{SP}_n$ is minimal with respect to $A \subseteq C, B \subseteq C$.

2. Decomposition of $\mathcal{E}_n(q)$

In this section we obtain central idempotents of $\mathcal{E}_n(q)$ and a corresponding subalgebra decomposition of $\mathcal{E}_n(q)$. This is inspired by I. Marin's recent paper [**32**], which in turn is inspired by [**43**] and [**17**].

Recall that for a finite poset $(\Gamma, \preceq)$ there is an associated Möebius function $\mu_\Gamma : \Gamma \times \Gamma \to \mathbb{Z}$. In our set partition case $(\mathcal{SP}_n, \subseteq)$ the Möebius function $\mu_{\mathcal{SP}_n}$ is given by the formula

$$\mu_{\mathcal{SP}_n}(A,B) = \begin{cases} (-1)^{r-s} \prod_{i=1}^{r-1} (i!)^{r_{i+1}} & \text{if } A \subseteq B \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

where r and s are the number of blocks of A and B respectively, and where r_i is the number of blocks of B containing exactly i blocks of A.

We use the Möebius function $\mu = \mu_{\mathcal{SP}_n}$ to introduce a set of orthogonal idempotents elements of $\mathcal{E}_n(q)$. This is a special case of the general construction given in *loc. cit.* For $A \in \mathcal{SP}_n$ the idempotent $\mathbb{E}_A \in \mathcal{E}_n(q)$ is given by the formula

$$\mathbb{E}_A := \sum_{A\subseteq B} \mu(A,B) E_B. \tag{2.2}$$

For example, we have $\mathbb{E}_{\{\{1\},\{2\},\{3\}\}} = E_{\{\{1\},\{2\},\{3\}\}} - E_{\{\{1,2\},\{3\}\}} - E_{\{\{1\},\{2,3\}\}} - E_{\{\{1,3\},\{2\}\}} + 2E_{\{\{1,2,3\}\}}$. We have the following result.

PROPOSITION 3.1. *The following properties hold.*

(1) $\{\mathbb{E}_A | A \in \mathcal{SP}_n\}$ *is a set of orthogonal idempotents of* $\mathcal{E}_n(q)$.

(2) *For all* $w \in \mathfrak{S}_n$ *and* $A \in \mathcal{SP}_n$ *we have* $\mathbb{E}_A g_w = g_w \mathbb{E}_{Aw}$.

(3) *For all* $A \in \mathcal{SP}_n$ *we have* $\mathbb{E}_A E_B = \begin{cases} \mathbb{E}_A & \text{if } B \subseteq A \\ 0 & \text{if } B \not\subseteq A. \end{cases}$

PROOF. We have already mentioned (1) so let us prove (2). We first note that the order relation $\subseteq$ on $\mathcal{SP}_n$ is compatible with the action of $\mathfrak{S}_n$ on $\mathcal{SP}_n$ that is $A \subseteq B$ if and only if $Aw \subseteq Bw$ for all $w \in \mathfrak{S}_n$. This implies that $\mu(Aw, Bw) = \mu(A,B)$ for all $w \in \mathfrak{S}_n$. From this we get, via (1.12), that

$$\mathbb{E}_A g_w = g_w \sum_{A\subseteq B} \mu(A,B) E_{Bw} = g_w \sum_{A\subseteq Cw^{-1}} \mu(A, Cw^{-1}) E_C = g_w \sum_{Aw\subseteq C} \mu(Aw, C) E_C = g_w \mathbb{E}_{Aw}$$

showing (2). Finally, we obtain (3) from the orthogonality of the $\mathbb{E}_A$'s and the formula $E_B = \sum_{B\subseteq A} \mathbb{E}_A$ which is obtained by inverting (2.2) (see also [**17**]). □

We say that a set partition $A = \{I_1, \dots, I_k\}$ of $\mathbf{n}$ is of type $\alpha \in \mathcal{P}ar_n$ if there exists a permutation σ such that $(|I_{i_{1\sigma}}|, \dots, |I_{i_{k\sigma}}|) = \alpha$. For example, the set partitions of $\mathbf{3}$ of type $(2,1)$ are $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$ and $\{\{2,3\},\{1\}\}$. For short, we write $|A| = \alpha$ if $A \in \mathcal{SP}_n$ is of type α. We also say that a multicomposition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in Comp_{r,n}$ is of type α if the associated set partition $A_{\boldsymbol{\lambda}}$ is of type α.

For each $\alpha \in \mathcal{P}ar_n$ we define the following element $\mathbb{E}_\alpha$ of E

$$\mathbb{E}_\alpha := \sum_{A\in\mathcal{SP}_n, |A|=\alpha} \mathbb{E}_A. \tag{2.3}$$

Then by Proposition 3.1 we have that $\{\mathbb{E}_\alpha | \alpha \in \mathcal{P}ar_n\}$ is a set of central orthogonal idempotents of $\mathcal{E}_n(q)$, which is complete: $\sum_{\alpha\in\mathcal{P}ar_n} \mathbb{E}_\alpha = 1$. As an immediate consequence we get the following decomposition of $\mathcal{E}_n(q)$ into a direct sum of two-sided ideals

$$\mathcal{E}_n(q) = \bigoplus_{\alpha\in\mathcal{P}ar_n} \mathcal{E}_n^\alpha(q) \tag{2.4}$$

where we define $\mathcal{E}_n^\alpha(q) := \mathbb{E}_\alpha \mathcal{E}_n(q)$. Each $\mathcal{E}_n^\alpha(q)$ is an S-algebra with identity $\mathbb{E}_\alpha$.

Using the $\{E_A g_w\}$-basis for $\mathcal{E}_n(q)$, together with part (3) of Proposition 3.1, we get that the set

$$\{\mathbb{E}_A g_w \mid w \in \mathfrak{S}_n, |A| = \alpha\} \tag{2.5}$$

is an S-basis for $\mathcal{E}_n^{\alpha}(q)$. In particular, we have that the dimension of $\mathcal{E}_n^{\alpha}(q)$ is $b_n(\alpha)n!$, where $b_n(\alpha)$ is the number of set partitions of $\mathbf{n}$ having type $\alpha \in \mathcal{P}ar_n$. The numbers $b_n(\alpha)$ are the socalled Faà di Bruno coefficients and are given by the following formula

$$b_n(\alpha) = \frac{n!}{(k_1!)^{m_1} m_1! \cdots (k_r!)^{m_r} m_r!} \tag{2.6}$$

where $\alpha = (k_1^{m_1}, \ldots, k_r^{m_r})$ and $k_1 > \ldots > k_r$.

3. Cellular basis for $\mathcal{E}_n(q)$

In the paper [**39**], the representation theory of $\mathcal{E}_n(q)$ was studied in the generic case, where a parametrizing set for the irreducible modules was found. On the other hand, the dimensions of the generically irreducible modules were not determined in that paper. In this section we show that $\mathcal{E}_n(q)$ is a cellular algebra by giving a concrete combinatorial construction of a cellular basis for it. As a bonus we obtain a closed formula for the dimensions of the cell modules, which in particular gives a formula for the irreducible modules in the generic case. Although the construction of the cellular basis for $\mathcal{E}_n(q)$ follows the outline of the construction of the cellular basis $\mathcal{B}_{r,n}$ for $\mathcal{Y}_{r,n}(q)$, the combinatorial details are quite a lot more involved and, as we shall see, involve a couple of new ideas.

It should be pointed out that Jacon and Poulain d'Andecy have recently obtained a very elegant classification of the irreducible modules for $\mathcal{E}_n(q)$ via Clifford theory, see [**22**]. Their approach relies on the connection with the Yokonuma-Hecke algebra and therefore does not work for all fields. Our cellular algebra approach works, at least in principle, for all fields.

Let us explain the ingredients of our cellular basis for $\mathcal{E}_n(q)$. The antiautomorphism $*$ is easy to explain, since one easily checks on the relations for $\mathcal{E}_n(q)$ that $\mathcal{E}_n(q)$ is endowed with an S-linear antiautomorphism $*$, satisfying $e_i^* := e_i$ and $g_i^* := g_i$. We have that $\mathbb{E}_A^* = \mathbb{E}_A$.

Next we explain the poset denoted Λ in Definition 1.1 of cellular algebras. By general principles, it should be the parametrizing set for the irreducible modules for $\mathcal{E}_n(q)$ in the generic situation, so let us therefore recall this set $\mathcal{L}_n$ from [**39**]. $\mathcal{L}_n$ is the set of pairs $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ where $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ is an m-multipartition of n. We require that $\boldsymbol{\lambda}$ be *increasing* by which we mean that $\lambda^{(i)} < \lambda^{(j)}$ only if $i < j$ where $<$ is any fixed extension of the usual dominance order on partitions to a total order, and where we set $\lambda < \tau$ if λ and τ are partitions such that $|\lambda| < |\tau|$.

In order to describe the $\boldsymbol{\mu}$-ingredient of Λ we need to introduce some more notation. The multiplicities of equal $\lambda^{(i)}$'s give rise to a composition of m. To be more precise, let m_1 be the maximal i such that $\lambda^{(1)} = \lambda^{(2)} = \ldots = \lambda^{(i)}$, let m_2 be the maximal i such that $\lambda^{(m_1+1)} = \lambda^{(m_1+2)} = \ldots = \lambda^{(m_1+i)}$, and so on until m_q. Then we have that $m = m_1 + \ldots + m_q$. We then require that $\boldsymbol{\mu}$ be of the form $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(q)})$ where each $\mu^{(i)}$ is partition of m_i. This is the description of $\mathcal{L}_n$ as a set, as given in [**39**]. If $\alpha \in \mathcal{P}ar_n$ we use the notation

$$\mathcal{L}_n(\alpha) := \Big\{ (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n \Big| \boldsymbol{\lambda} \text{ is of type } \alpha \Big\}. \tag{3.1}$$

We now introduce a poset structure on $\mathcal{L}_n$. Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ and $\overline{\Lambda} = (\overline{\boldsymbol{\lambda}} \mid \overline{\boldsymbol{\mu}})$ are elements of $\mathcal{L}_n$ such that $\|\boldsymbol{\lambda}\| = \left\|\overline{\boldsymbol{\lambda}}\right\|$. We first write $\boldsymbol{\lambda} \rhd_1 \overline{\boldsymbol{\lambda}}$ if $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ and $\overline{\boldsymbol{\lambda}} = (\overline{\lambda^{(1)}}, \dots, \overline{\lambda^{(m)}})$ and if there exists a permutation σ such that $(\lambda^{(1\sigma)}, \dots, \lambda^{(m\sigma)}) \rhd (\overline{\lambda^{(1)}}, \dots, \overline{\lambda^{(m)}})$ where $\rhd$ is the dominance order on m-multipartitions, introduced above. We then say that $\Lambda \rhd \overline{\Lambda}$ if $\boldsymbol{\lambda} \rhd_1 \overline{\boldsymbol{\lambda}}$ or if $\boldsymbol{\lambda} = \overline{\boldsymbol{\lambda}}$ and $\boldsymbol{\mu} \rhd \overline{\boldsymbol{\mu}}$. As usual we set $\Lambda \unrhd \overline{\Lambda}$ if $\Lambda \rhd \overline{\Lambda}$ or if $\Lambda = \overline{\Lambda}$. This is our description of $\mathcal{L}_n$ as a poset. Note that if $\|\boldsymbol{\lambda}\| \neq \left\|\overline{\boldsymbol{\lambda}}\right\|$ then Λ and $\overline{\Lambda}$ are by definition not comparable.

REMARK 8. *We could have introduced an order '$\succ$' on $\mathcal{L}_n$ by replacing '$\rhd_1$' by '$\rhd$' in the above definition, that is $\Lambda \succ \overline{\Lambda}$ if $\boldsymbol{\lambda} \rhd \overline{\boldsymbol{\lambda}}$ or if $\boldsymbol{\lambda} = \overline{\boldsymbol{\lambda}}$ and $\boldsymbol{\mu} \rhd \overline{\boldsymbol{\mu}}$. Then '$\succ$' is a finer order than '$\rhd$', but in general they are different. The reason why we need to work with '$\rhd$' rather than '$\succ$' comes from the straightening procedure of Lemma 25 below.*

We could also have introduced an order on $\mathcal{L}_n$ by replacing '$=$' with '$=_1$' in the above definition, where '$=_1$' is defined via a permutation σ, similar to what we did for $\rhd_1$: that is $\Lambda \succ \overline{\Lambda}$ if $\boldsymbol{\lambda} \rhd_1 \overline{\boldsymbol{\lambda}}$ or if $\boldsymbol{\lambda} =_1 \overline{\boldsymbol{\lambda}}$ and $\boldsymbol{\mu} \rhd \overline{\boldsymbol{\mu}}$. On the other hand, since $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ are assumed to be increasing multipartitions, we get that '$=_1$' is just usual equality '$=$' and hence we would get the same order on $\mathcal{L}_n$.

Let us give an example to illustrate our order.

EXAMPLE 3. *We first note that* $(3,3,1) \rhd (3,2,2)$ *in the dominance order on partitions, but both are incomparable with the partition* $(4,1,1,1)$. *Suppose now that* $(3,2,2) < (4,1,1,1) < (3,3,1)$ *in our extension of the dominance order. We then consider the following increasing multipartitions of 25*

$$\boldsymbol{\lambda} = ((2),(2),(3,2,2),(4,1,1,1),(3,3,1)) \ \ \textit{and} \ \ \overline{\boldsymbol{\lambda}} = ((2),(2),(3,2,2),(3,2,2),(4,1,1,1)).$$

Then we have that $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ are increasing multipartitions, but incomparable in the dominance order on multipartitions. On the other hand $\boldsymbol{\lambda} \rhd_1 \overline{\boldsymbol{\lambda}}$ via the permutation $\sigma = s_4$ and hence we have the following relation in $\mathcal{L}_n$

$$\Lambda := \Big(\boldsymbol{\lambda} \,\Big|\, \big((2),(1),(1),(1)\big)\Big) \rhd \Big(\overline{\boldsymbol{\lambda}} \,\Big|\, \big((1^2),(2),(1)\big)\Big) =: \overline{\Lambda}.$$

For $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n$ as above, we next define the concept of Λ-tableaux. Suppose that $\mathbb{t}$ is a pair $\mathbb{t} = (\mathbf{t} \mid \mathbf{u})$. Then $\mathbb{t}$ is called a Λ-tableau if $\mathbf{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(m)})$ is a multitableau of n in the usual sense, satisfying $Shape(\mathbf{t}) = \boldsymbol{\lambda}$, and $\mathbf{u}$ is a $\boldsymbol{\mu}$-multitableau of the initial kind. As usual, if $\mathbb{t}$ is Λ-tableau we define $Shape(\mathbb{t}) := \Lambda$.

Let $\mathrm{Tab}(\Lambda)$ denote the set of Λ-tableaux and let $\mathrm{Tab}_n := \cup_{\Lambda \in \mathcal{L}_n} \mathrm{Tab}(\Lambda)$. We then say that $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}) \in \mathrm{Tab}(\Lambda)$ is row standard if its ingredients are row standard multitableaux in the usual sense.

We say that $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}) \in \mathrm{Tab}(\Lambda)$ is standard if its ingredients are standard multitableaux and if moreover $\mathbf{t}$ is an *increasing* multitableau. By increasing we here mean that whenever $\lambda^{(i)} = \lambda^{(j)}$ we have that $i < j$ if and only if $\min(\mathfrak{t}^{(i)}) < \min(\mathfrak{t}^{(j)})$ where $\min(t)$ is the function that reads off the minimal entry of the tableau t. We define $\mathrm{Std}(\Lambda)$ to be the set of all standard Λ-tableaux.

EXAMPLE 4. *For* $\Lambda = \Big(\big((1,1),(2),(2),(2,1)\big) \,\big|\, \big((1),(1,1),(1)\big)\Big)$ *we consider the following* Λ*-tableaux*

$$\begin{aligned}\mathbb{t}_1 &:= \left(\left(\begin{smallmatrix}\boxed{1}\\\boxed{9}\end{smallmatrix}, \boxed{3}\boxed{5}, \boxed{6}\boxed{8}, \begin{smallmatrix}\boxed{2}\boxed{4}\\\boxed{7}\end{smallmatrix}\right) \,\Big|\, \left(\boxed{1}, \begin{smallmatrix}\boxed{2}\\\boxed{3}\end{smallmatrix}, \boxed{4}\right)\right)\\ \mathbb{t}_2 &:= \left(\left(\begin{smallmatrix}\boxed{1}\\\boxed{9}\end{smallmatrix}, \boxed{5}\boxed{6}, \boxed{3}\boxed{8}, \begin{smallmatrix}\boxed{2}\boxed{4}\\\boxed{7}\end{smallmatrix}\right) \,\Big|\, \left(\boxed{1}, \begin{smallmatrix}\boxed{2}\\\boxed{3}\end{smallmatrix}, \boxed{4}\right)\right).\end{aligned} \tag{3.2}$$

Then by our definition, $\mathbb{t}_1$ *is a standard* Λ*-tableau, but* $\mathbb{t}_2$ *is not.*

REMARK 9. *The use of the function* $\min(\cdot)$ *is somewhat arbitrary. In fact we could have used any injective function with values in a totally ordered set.*

For $\mathbf{t} = (\mathrm{t}^{(1)},\dots,\mathrm{t}^{(m)})$ and $\overline{\mathbf{t}} = (\overline{\mathrm{t}^{(1)}},\dots,\overline{t^{(m)}})$ we define $\mathbf{t} \rhd_1 \overline{\mathbf{t}}$ if there exists a permutation σ such that $(\mathrm{t}^{(1\sigma)},\dots,\mathrm{t}^{(m\sigma)}) \rhd (\overline{\mathrm{t}^{(1)}},\dots,\overline{\mathrm{t}^{(m)}})$ in the sense of multitableaux. We then extend the order on $\mathcal{L}_n$ to Tab_n as follows. Suppose that $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}) \in \mathrm{Tab}(\Lambda)$ and $\overline{\mathbb{t}} = (\overline{\mathbf{t}} \mid \overline{\mathbf{u}}) \in \mathrm{Tab}(\overline{\Lambda})$ and that $\Lambda \unrhd \overline{\Lambda}$. Then we say that $\mathbf{t} \rhd \overline{\mathbf{t}}$ if $\mathbf{t} \rhd_1 \overline{\mathbf{t}}$ or if $\mathbf{t} = \overline{\mathbf{t}}$ and $\mathbf{u} \rhd \overline{\mathbf{u}}$. As usual we set $\mathbb{t} \unrhd \overline{\mathbb{t}}$ if $\mathbb{t} \rhd \overline{\mathbb{t}}$ or $\mathbb{t} = \overline{\mathbb{t}}$. This finishes our description of Λ-tableaux as a poset.

From the basis of $\mathcal{E}_n(q)$ mentioned above, we have that $\dim \mathcal{E}_n(q) = b_n n!$ where b_n is the n'th Bell number, that is the number of set partitions on $\mathbf{n}$. Our next Lemma is a first strong indication of the relationship between our notion of standard tableaux and the representation theory of $\mathcal{E}_n(q)$.

Recall the notation $d_\lambda := |\mathrm{Std}(\lambda)|$ that we introduced for partitions λ. In the proof of the Lemma, and later on, we shall use repeatedly the formula $\sum_{\lambda \in \mathcal{P}ar_n} d_\lambda^2 = n!$.

LEMMA 18. *With the above notation we have that* $\sum_{\Lambda \in \mathcal{L}_n} |\mathrm{Std}(\Lambda)|^2 = b_n n!$.

PROOF. It is enough to prove the formula

$$\sum_{\Lambda \in \mathcal{L}_n(\alpha)} |\mathrm{Std}(\Lambda)|^2 = b_n(\alpha) n! \tag{3.3}$$

where $b_n(\alpha)$ is the Faà di Bruno coefficient introduced above. Let us first consider the case $\alpha = (k^m)$, that is $n = mk$. Then we have

$$b_n(m,k) := b_n(\alpha) = \frac{1}{m!}\binom{n}{k\cdots k}$$

with k appearing m times in the multinomial coefficient. Let $\{\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)}\}$ be the fixed ordered enumeration of all the partitions of k, introduced above. If $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$ then $\boldsymbol{\lambda}$ has the form

$$\boldsymbol{\lambda} = (\overbrace{\lambda^{(1)},\dots,\lambda^{(1)}}^{m_1}, \overbrace{\lambda^{(2)},\dots,\lambda^{(2)}}^{m_2}, \dots, \overbrace{\lambda^{(d)}\dots,\lambda^{(d)}}^{m_d})$$

where the m_i's are non-negative integers with sum m and $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(d)})$ is a multipartition of type $\|\boldsymbol{\mu}\| = (m_1, m_2, \ldots, m_d)$. The number of increasing multitableaux of shape $\boldsymbol{\lambda}$ is

$$\frac{1}{m_1! \ldots m_d!} \binom{n}{k \cdots k} \prod_{j=1}^{d} d_{\lambda^{(j)}}^{m_j}$$

whereas the number of standard tableaux of shape $\boldsymbol{\mu}$ is $\prod_{j=1}^{d} d_{\mu^{(j)}}$ and so we get

$$|\mathrm{Std}(\Lambda)| = \frac{1}{m!} \binom{m}{m_1 \cdots m_d} \binom{n}{k \cdots k} \prod_{j=1}^{d} d_{\lambda^{(j)}}^{m_j} d_{\mu^{(j)}} \tag{3.4}$$

By first fixing $\boldsymbol{\lambda}$ and then letting each $\mu^{(i)}$ vary over all possibilities we get that the square sum of the above $|\mathrm{Std}(\Lambda)|$'s is the sum of

$$\binom{n}{k \cdots k}^2 \prod_{j=1}^{d} \frac{d_{\lambda^{(j)}}^{2m_j}}{m_j!} = \binom{n}{k \cdots k}^2 \frac{1}{m!} \binom{m}{m_1 \cdots m_d} \prod_{j=1}^{d} d_{\lambda^{(j)}}^{2m_j}$$

with the m_i's running over the above mentioned set of numbers. But by the multinomial formula, this sum is equal to

$$\binom{n}{k \cdots k}^2 \frac{1}{m!} \left(\sum_{j=1}^{d} d_{\lambda^{(j)}}^2 \right)^m = \binom{n}{k \cdots k}^2 \frac{1}{m!} k!^m = \frac{n!}{m!} \binom{n}{k \cdots k} = b_n(\alpha) n!$$

and (3.3) is proved in this case.

Let us now consider the general case where $\alpha = (k_1^{M_1}, \ldots, k_r^{M_r})$, where $k_1 > \cdots > k_r$. Set $n_i = k_i M_i$, $M := M_1 + \ldots + M_r$. Then $n = n_1 + \ldots + n_r$ and the Faà di Bruno coefficient $b_n(\alpha)$ is given by the formula

$$b_n(\alpha) = \binom{n}{n_1 \cdots n_r} b_{n_1}(M_1, k_1) \cdots b_{n_r}(M_r, k_r). \tag{3.5}$$

Let us now consider the square sum $\sum_{\Lambda \in \mathcal{L}_n(\alpha)} |\mathrm{Std}(\Lambda)|^2$. For $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$ we split $\boldsymbol{\lambda}$ into multipartitions $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_r$, where $\boldsymbol{\lambda}_1 = (\lambda^{(1)}, \ldots, \lambda^{(M_1)})$, $\boldsymbol{\lambda}_2 = (\lambda^{(M_1+1)}, \ldots, \lambda^{(M_1+M_2)})$, and so on. We split $\boldsymbol{\mu}$ correspondingly into $\boldsymbol{\mu}_i$'s and set $\Lambda_i := (\boldsymbol{\lambda}_i \mid \boldsymbol{\mu}_i)$. Then $\Lambda_i \in \mathcal{L}_{n_i}((k_i^{M_i}))$ and we have

$$|\mathrm{Std}(\Lambda)| = \binom{n}{n_1 \cdots n_r} |\mathrm{Std}_{n_1}(\Lambda_1)| \cdots |\mathrm{Std}_{n_r}(\Lambda_r)| \tag{3.6}$$

where $\mathrm{Std}_{n_i}(\Lambda_i)$ means standard tableaux of shape Λ_i on $\mathbf{n_i}$. Combining (3.3), (3.5) and (3.6) we get that

$$\sum_{\Lambda \in \mathcal{L}_n(\alpha)} |\mathrm{Std}(\Lambda)|^2 = n! b_n(\alpha)$$

as claimed. □

COROLLARY 3.1. *Suppose that* $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n$ *is above with* $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ *and* $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(q)})$ *and set* $n_i := |\lambda^{(i)}|$ *and* $m_i := |\mu^{(i)}|$. *Then we have that*

$$|\mathrm{Std}(\Lambda)| = \frac{1}{m_1! \cdots m_q!} \binom{n}{n_1 \cdots n_m} \prod_{j=1}^{m} d_{\lambda^{(j)}} \prod_{j=1}^{q} d_{\mu^{(j)}}.$$

PROOF. This follows by combining (3.4) and (3.6) from the proof of the Lemma. $\square$

We fix the following combinatorial notation. Let $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) = ((\lambda^{(1)},\dots,\lambda^{(m)}) \mid (\mu^{(1)},\dots,\mu^{(q)})) \in \mathcal{L}_n(\alpha)$. With Λ we have associated the set of multiplicities $\{m_i\}_{i=1,\dots,q}$ of equal $\lambda^{(i)}$'s. We now also associate with Λ the set of multiplicities $\{k_i\}_{i=1,\dots,r}$ of equal block sizes $|\lambda^{(i)}|$. That is, k_1 is the maximal i such that $|\lambda^{(1)}| = |\lambda^{(2)}| = \ldots = |\lambda^{(i)}|$, whereas k_2 is the maximal i such that $|\lambda^{(k_1+1)}| = |\lambda^{(k_1+2)}| = \ldots = |\lambda^{(k_1+i)}|$ and so on. We can also describe the k_i's in terms of the type of $\boldsymbol{\lambda}$, that is α: indeed we have $\alpha = (a_r^{k_r},\dots,a_1^{k_1})$ with $a_r > a_{r-1} > \ldots > a_1$: recall that $\boldsymbol{\lambda}$ is increasing. Note that $m_1 + m_2 + \cdots + m_q = k_1 + k_2 + \cdots + k_r = m$ and that $|\mu^{(j)}| = m_j$ for all j.

Let $\mathfrak{S}_\Lambda \le \mathfrak{S}_n$ be the stabilizer subgroup of the set partition $A_{\boldsymbol{\lambda}} = \{I_1, I_2, \dots, I_m\}$ that was introduced in (3.2). Then the two sets of multiplicities give rise to subgroups $\mathfrak{S}_\Lambda^k$ and $\mathfrak{S}_\Lambda^m$ of $\mathfrak{S}_\Lambda$ where $\mathfrak{S}_\Lambda^k$ consists of the order preserving permutations of the equally sized blocks of $A_{\boldsymbol{\lambda}}$, whereas $\mathfrak{S}_\Lambda^m$ consists of the order preserving permutations of those blocks of $A_{\boldsymbol{\lambda}}$ that correspond to equal $\lambda^{(i)}$'s. Clearly we have $\mathfrak{S}_\Lambda^m \le \mathfrak{S}_\Lambda^k \le \mathfrak{S}_\Lambda$.

We observe that $\mathfrak{S}_\Lambda^k$ and $\mathfrak{S}_\Lambda^m$ are products of symmetric groups,

$$\mathfrak{S}_\Lambda^k \cong \mathfrak{S}_{k_1} \times \ldots \times \mathfrak{S}_{k_r}, \quad \mathfrak{S}_\Lambda^m \cong \mathfrak{S}_{m_1} \times \ldots \times \mathfrak{S}_{m_q} \tag{3.7}$$

and in fact $\mathfrak{S}_\Lambda^k$ is a Coxeter group on generators B_i that we explain shortly, and $\mathfrak{S}_\Lambda^m$ is a parabolic subgroup of $\mathfrak{S}_\Lambda^k$. Define subsets $S_\Lambda^m \subseteq S_\Lambda^k$ of $\mathbf{m}$ via

$$S_\Lambda^k := \{i \in \mathbf{m} | i \ne k_1 + \ldots + k_j \text{ for all } j\}, \quad S_\Lambda^m := \{i \in \mathbf{m} | i \ne m_1 + \ldots + m_j \text{ for all } j\}. \tag{3.8}$$

Then for $i \in S_\Lambda^k$ the generator B_i of $\mathfrak{S}_\Lambda^k$ is the minimal length element of $\mathfrak{S}_n$ that interchanges the two consecutive blocks I_i and I_{i+1} of $A_{\boldsymbol{\lambda}}$ (of equal size). Moreover, B_i is also a generator for $\mathfrak{S}_\Lambda^m$ if and only if $i \in \mathfrak{S}_\Lambda^m$. Let us describe B_i concretely. Letting $a := |I_i|$ we can write

$$I_i = \{c+1, c+2, \dots, c+a\} \text{ and } I_{i+1} = \{c+a+1, c+a+2, \dots, c+2a\} \tag{3.9}$$

for some c. With this notation we have

$$B_i = (c+1, c+a+1)(c+2, c+a+2)\cdots(c+a, c+2a). \tag{3.10}$$

For $i > j$ we set $s_{ij} := s_{i+c}s_{i-1+c}\dots s_{j+c}$ and can then write B_i in terms of the s_{ij}'s, and therefore in terms of simple transpositions s_i, as follows

$$B_i = s_{a,1}s_{a+1,2}\dots s_{2a-1,a}. \tag{3.11}$$

Our next step is to show that the group algebras $S\mathfrak{S}_\Lambda^m$ and $S\mathfrak{S}_\Lambda^k$ can be viewed as subalgebras of $\mathcal{E}_n(q)$. For this purpose and inspired by the formula (3.11) for B_i, we define $\mathbb{B}_i \in \mathcal{E}_n^\alpha(q)$ as follows

$$\mathbb{B}_i := \mathbb{E}_\Lambda g_{a,1}g_{a+1,2}\dots g_{2a-1,a}, \; g_{ij} := g_{i+c}g_{i-1+c}\cdots g_{j+c} \tag{3.12}$$

where we from now on use the notation

$$\mathbb{E}_\Lambda := \mathbb{E}_{A_{\boldsymbol{\lambda}}}. \tag{3.13}$$

We can now state our next result.

LEMMA 19. *Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$. Then we have S-algebra embeddings*

(1) $\iota : S\mathfrak{S}^m_\Lambda \hookrightarrow \mathcal{E}^\alpha_n(q)$, *via* $B_i \mapsto \mathbb{B}_i$ *for* $i \in S^m_\Lambda$.

(2) $\iota : S\mathfrak{S}^k_\Lambda \hookrightarrow \mathcal{E}^\alpha_n(q)$, *via* $B_i \mapsto \mathbb{B}_i$ *for* $i \in S^k_\Lambda$.

PROOF. It is enough to prove part (2) of the Lemma since $\mathfrak{S}^m_\Lambda$ is simply the parabolic subgroup of $\mathfrak{S}^k_\Lambda$ corresponding to S^m_Λ. Now $\boldsymbol{\lambda}$ is of type α and so the presence of the factor $\mathbb{E}_\Lambda$ in $\mathbb{B}_i$ gives via (2.5) that $\mathbb{B}_i \in \mathcal{E}^\alpha_n(q)$. Hence, in order to show the Lemma we need to check the following three identities

$$
\begin{aligned}
&a) \quad \mathbb{B}_i\mathbb{B}_{i+1}\mathbb{B}_i = \mathbb{B}_{i+1}\mathbb{B}_i\mathbb{B}_{i+1} \ \text{ for } \ i, i+1 \in S^k_\Lambda. \\
&b) \quad \mathbb{B}_i^2 = \mathbb{E}_\Lambda \ \text{ for } \ i \in S^k_\Lambda. \\
&c) \quad \mathbb{B}_i\mathbb{B}_j = \mathbb{B}_j\mathbb{B}_i \ \text{ for } \ |i-j|>1 \ \text{ and } \ i,j \in S^k_\Lambda.
\end{aligned}
\tag{3.14}
$$

Let us first take a closer look at the expansion $B_i = s_{i_1}s_{i_2}\cdots s_{i_p}$ according to the definitions of B_i and s_{ij}. We claim that this expansion is a reduced expression in the s_i's. Indeed, let $\underline{i} := (1,2,\ldots,n) \in \mathrm{seq}_n$. Then the right action of B_i on $\underline{i}$, according to $B_i = s_{i_1}s_{i_2}\cdots s_{i_p}$, changes at the j'th step $\ldots i_p \ldots i_p+1\ldots$ to $\ldots i_p+1\ldots i_p\ldots$, as one easily checks, and from this we conclude, via the inversion description of the length function on $\mathfrak{S}_n$, that $s_{i_1}s_{i_2}\cdots s_{i_p}$ indeed is a reduced expression for B_i. On the other hand, by the description of B_i in (3.10) we also have that $B_i = s_{i_p}\cdots s_{i_2}s_{i_1}$. By length considerations this must be a reduced expression for B_i as well, and hence via Matsumuto's Theorem we get that

$$\mathbb{B}_i = \mathbb{E}_\Lambda g_{i_p}\cdots g_{i_2}g_{i_1} \tag{3.15}$$

since, after all, the g_i's verify the braid relations.

In order to show a) and c), we now first observe, acting once again on the sequence $\underline{i}$ above, that the expansions of each side of these identities in terms of s_i's are also reduced expressions. On the other hand, by Proposition 3.1(2) we can commute $\mathbb{E}_\Lambda$ to the right of $\mathbb{B}_i$ that is $\mathbb{B}_i := g_{a,1}g_{a+1,2}\cdots g_{2a-1,a}\mathbb{E}_\Lambda$ and so we get a) and c) using Matsumuto's Theorem directly on the corresponding reduced expressions.

In order to show b) we have to argue a bit differently. It is enough to show that

$$\mathbb{B}_i^2 = \mathbb{E}_\Lambda g_{i_1}\cdots g_{i_{p-1}}g_{i_p}\mathbb{E}_\Lambda g_{i_p}g_{i_{p-1}}\cdots g_{i_1} = \mathbb{E}_\Lambda \tag{3.16}$$

since we can use (3.15) for the second expression for $\mathbb{B}_i$. Commuting g_{i_p} past $\mathbb{E}_\Lambda$ this becomes

$$\mathbb{E}_\Lambda g_{i_1}\cdots g_{i_{p-1}}g_{i_p}^2\mathbb{E}_{(A_{\boldsymbol{\lambda}})s_{i_p}}g_{i_{p-1}}\cdots g_{i_1} = \mathbb{E}_\Lambda g_{i_1}\cdots g_{i_{p-1}}\left(1+(q-q^{-1})g_{i_p}e_{i_p}\right)\mathbb{E}_{(A_{\boldsymbol{\lambda}})s_{i_p}}g_{i_{p-1}}\cdots g_{i_1} \tag{3.17}$$

by Proposition 3.1(2). But i_i and i_p+1 are in different blocks of $(A_{\boldsymbol{\lambda}})s_{i_p}$ and so we have $e_{i_p}\mathbb{E}_{(A_{\boldsymbol{\lambda}})s_{i_p}} = E_{i_p}\mathbb{E}_{(A_{\boldsymbol{\lambda}})s_{i_p}} = 0$ by Proposition 3.1(3). Hence (3.17) is equal to

$$\mathbb{E}_\Lambda g_{i_1}\cdots g_{i_{p-1}}\mathbb{E}_{(A_{\boldsymbol{\lambda}})s_{i_p}}g_{i_{p-1}}\cdots g_{i_1}. \tag{3.18}$$

With the same reasoning we move $g_{i_{p-1}}$ past $\mathbb{E}_{(A_\lambda)s_{i_p}}$ to arrive at

$$\mathbb{E}_\Lambda g_{i_1} \cdots g_{i_{p-2}} \mathbb{E}_{(A_\lambda)s_{i_p}s_{i_{p-1}}} g_{i_{p-2}} \cdots g_{i_1} \tag{3.19}$$

and so on until

$$\mathbb{E}_\Lambda \mathbb{E}_{(A_\lambda)s_{i_p}s_{i_{p-1}}\dots s_{i_1}} = \mathbb{E}_\Lambda \mathbb{E}_{A_\lambda} = \mathbb{E}_\Lambda \tag{3.20}$$

via Proposition 3.1(2). This proves b).

For B_y an element of $\mathfrak{S}_\Lambda^k$ written in reduced form as $B_y = B_{i_1} B_{i_2} \cdots B_{i_k}$ with $i_j \in S_\Lambda^k$, we define

$$\mathbb{B}_y := \mathbb{B}_{i_1} \mathbb{B}_{i_2} \cdots \mathbb{B}_{i_k} \in \mathcal{E}_n^\alpha(q). \tag{3.21}$$

Then by the above, $\mathbb{B}_y \in \mathcal{E}_n^\alpha(q)$ is independent of the chosen reduced expression. Since $\mathbb{E}_\Lambda$ commutes with $\iota(S\mathfrak{S}_\Lambda^k)$ and since $\iota(B_i) = \mathbb{B}_i$ we have that $\iota(B_y) = \mathbb{B}_y$.

It only remains to show that the induced homomorphism $\iota : S\mathfrak{S}_\Lambda^k \to \mathcal{E}_n^\alpha(q)$ is an enbedding. But this follows directly from the basis $\mathcal{B} := \{\mathbb{E}_A g_w\}$ for $\mathcal{E}_n^\alpha(q)$ given in (2.5). Indeed, we have that $\iota(B_y) = \mathbb{B}_y = \mathbb{E}_\Lambda g_y \in \mathcal{B}$ where $y \in \mathfrak{S}_n$ is the element obtained by expanding $B_y = B_{i_1} B_{i_2} \cdots B_{i_k}$ completely in terms of s_i's. From this it also follows that $\iota(B_y) = \iota(B_w)$ iff $B_y = B_w$, proving the injectivity of ι. □

REMARK 10. *The identity element of $\iota(S\mathfrak{S}_\Lambda^k)$ is $\mathbb{E}_\Lambda$ whereas the identity element of $\mathcal{E}_n^\alpha(q)$ is $\mathbb{E}_\alpha$, as was seen implicitly in the proof. In particular, ι does not preserve identity elements.*

Suppose that $y \in \mathfrak{S}_\Lambda^k$ and let $y := s_{i_1} \dots s_{i_k}$ be a reduced expression. Then we define $B_y := B_{i_1} \dots B_{i_k}$ and $\mathbb{B}_y := \mathbb{B}_{i_1} \dots \mathbb{B}_{i_k}$. Note that, by the above Lemma, $\mathbb{B}_y \in \mathcal{E}_n(q)$ is independent of the chosen reduced expression.

Recall that for any S-algebra $\mathcal{A}$, the wreath product algebra $\mathcal{A} \wr \mathfrak{S}_f$ is defined as the semidirect product $\mathcal{A}^{\otimes f} \rtimes \mathfrak{S}_f$ where $\mathfrak{S}_f$ acts on $\mathcal{A}^{\otimes f}$ via place permutation. If $\mathcal{A}$ is free over S with basis B then $\mathcal{A} \wr \mathfrak{S}_f$ is also free over S with basis $(b_{i_1} \otimes \cdots \otimes b_{i_f}) \otimes w$ where $b_{i_j} \in B$ and where $w \in \mathfrak{S}_f$. There are canonical algebra embeddings $i_{\mathcal{A},f} : \mathcal{A}^{\otimes f} \hookrightarrow \mathcal{A} \wr \mathfrak{S}_f$ and $j_{\mathcal{A},f} : S\mathfrak{S}_f \hookrightarrow \mathcal{A} \wr \mathfrak{S}_f$ whose images generate $\mathcal{A} \wr \mathfrak{S}_f$, subject to the following relations

$$j_{\mathcal{A},f}(w) i_{\mathcal{A},f}(b_{i_1} \otimes \cdots \otimes b_{i_f}) = i_{\mathcal{A},f}(b_{i_{1w^{-1}}} \otimes \cdots \otimes b_{i_{fw^{-1}}}) j_{\mathcal{A},f}(w). \tag{3.22}$$

Recall $\mathfrak{S}_\Lambda \leq \mathfrak{S}_n$, the stabilzer subgroup of the set partition A_λ. With the above notation we have an isomorphism

$$S\mathfrak{S}_\Lambda \cong S\mathfrak{S}_{a_1} \wr \mathfrak{S}_{k_1} \otimes \cdots \otimes S\mathfrak{S}_{a_r} \wr \mathfrak{S}_{k_r}. \tag{3.23}$$

We are interested in the following deformation of $S\mathfrak{S}_\Lambda$

$$\mathcal{H}_\alpha^{wr}(q) := \mathcal{H}_{a_1}(q) \wr \mathfrak{S}_{k_1} \otimes \cdots \otimes \mathcal{H}_{a_r}(q) \wr \mathfrak{S}_{k_r}. \tag{3.24}$$

Recall that we have $S\mathfrak{S}_\Lambda^k = S\mathfrak{S}_{k_1} \otimes \cdots \otimes S\mathfrak{S}_{k_r}$ by (3.7). Let

$$j : S\mathfrak{S}_\Lambda^k = S\mathfrak{S}_{k_1} \otimes \cdots \otimes S\mathfrak{S}_{k_r} \hookrightarrow \mathcal{H}_\alpha^{wr}(q) \tag{3.25}$$

be the embedding induced by the $j_{\mathcal{H}_{a_i},k_i}$'s and let

$$i : \mathcal{H}_{a_1}(q)^{\otimes k_1} \otimes \cdots \otimes \mathcal{H}_{a_r}(q)^{\otimes k_r} \hookrightarrow \mathcal{H}_\alpha^{wr}(q) \tag{3.26}$$

be the embedding induced by the $i_{\mathfrak{S}_{a_i},k_i}$'s.

Now $\mathcal{H}_{a_1}(q)^{\otimes k_1} \otimes \cdots \otimes \mathcal{H}_{a_r}(q)^{\otimes k_r}$ is canonically isomorphic to the Young-Hecke algebra $\mathcal{H}_{\alpha^{op}}(q)$. Moreover, the multiplication map $g_{w_1} \cdots g_{w_m} \mapsto \mathbb{E}_\Lambda g_{w_1} \cdots g_{w_m}$ induces an embedding of $\mathcal{H}_{\alpha^{op}}(q)$ in $\mathcal{E}_n^\alpha(q)$. Combining, and using the basis (2.6), we get an embedding

$$\epsilon : \mathcal{H}_{a_1}(q)^{\otimes k_1} \otimes \cdots \otimes \mathcal{H}_{a_r}(q)^{\otimes k_r} \hookrightarrow \mathcal{E}_n^\alpha(q). \tag{3.27}$$

With these preparations we can now extend Lemma 19 to $\mathcal{H}_\alpha^{wr}(q)$ as follows.

LEMMA 20. *There is a unique embedding*

$$\upsilon : \mathcal{H}_\alpha^{wr}(q) \hookrightarrow \mathcal{E}_n^\alpha(q) \tag{3.28}$$

such that $\epsilon : \mathcal{H}_{a_1}(q)^{\otimes k_1} \otimes \cdots \otimes \mathcal{H}_{a_r}(q)^{\otimes k_r} \hookrightarrow \mathcal{E}_n^\alpha(q)$ *factorizes as* $\epsilon = \upsilon \circ i$ *and such that* $\iota : S\mathfrak{S}_\Lambda^k \hookrightarrow \mathcal{E}_n^\alpha(q)$ *from the previous Lemma factorizes as* $\iota = \upsilon \circ j$.

PROOF. Let $\epsilon_i : \mathcal{H}_{a_i}(q)^{\otimes k_i} \to \mathcal{E}_n^\alpha(q)$ be the composition of the canonical embedding

$$\mathcal{H}_{a_i}(q)^{\otimes k_i} \hookrightarrow \mathcal{H}_{a_1}(q)^{\otimes k_1} \otimes \cdots \otimes \mathcal{H}_{a_i}(q)^{\otimes k_i} \otimes \cdots \otimes \mathcal{H}_{a_r}(q)^{\otimes k_r}$$

with ϵ and let $\iota_i : S\mathfrak{S}_{k_i} \to \mathcal{E}_n^\alpha(q)$ be the composition of the canonical embedding

$$S\mathfrak{S}_{k_i} \longrightarrow S\mathfrak{S}_{k_1} \otimes \cdots \otimes S\mathfrak{S}_{k_i} \otimes \cdots \otimes S\mathfrak{S}_{k_r} = S\mathfrak{S}_\Lambda^k \longrightarrow \mathcal{H}_\alpha^{wr}(q)$$

with ι. The existence and uniqueness of υ follows from the universal property of the wreath product. In other words, by (3.22) we must check that

$$\iota_i(w)\epsilon_i(g_{y_1} \otimes \cdots \otimes g_{y_{k_i}}) = \epsilon_i(g_{y_{1w^{-1}}} \otimes \cdots \otimes g_{y_{k_i w^{-1}}})\iota_i(w) \tag{3.29}$$

where $w \in \mathfrak{S}_{k_i}$ and the g_{y_j}'s belong to $\mathcal{H}_{a_i}(q)$. By the definitions, this becomes the following equality in $\mathcal{E}_n^\alpha(q)$

$$\mathbb{B}_w g_{y_1} \cdots g_{y_{k_i}} = g_{y_{1w^{-1}}} \cdots g_{y_{k_i w^{-1}}} \mathbb{B}_w \tag{3.30}$$

where the g_{y_j}'s belong to the distinct Hecke algebras given by the k_i distinct Hecke algebra factors of $\epsilon_i(\mathcal{H}_{a_i}(q)^{\otimes k_i})$ and similarly for the $g_{y_{jw^{-1}}}$'s.

To verify this we may assume $i = 1$ and $r = 1$. Let $k_1 := k$ and $a_1 := a$. Assume that $g_{y_1} := g_s$ with $s \in \{1, \ldots, ka\}$ and $a \nmid s$ and let $\mathbb{B}_w = \mathbb{B}_j$ where $1 \le j < k$. Then (3.30) reduces to proving

$$\begin{array}{lll} a)\ \mathbb{B}_j g_s = g_{s+a}\mathbb{B}_j & \text{if } s \in \{(j-1)a+1, (j-1)a+2, \ldots, ja-1\} \\ b)\ \mathbb{B}_j g_s = g_{s-a}\mathbb{B}_j & \text{if } s \in \{ja+1, j+2, \ldots, ja+a-1\} \\ c)\ \mathbb{B}_j g_s = g_s\mathbb{B}_j & \text{otherwise.} \end{array} \tag{3.31}$$

Let us assume that $j = 1$, the other cases are treated similarly. Then in the notation of (3.10) we have that $c = 0$ and by (3.12) $g_{ij} := g_i g_{i-1} \cdots g_j$ and

$$\mathbb{B}_i := \mathbb{E}_\Lambda g_{a,1} g_{a+1,2} \cdots g_{2a-1,a}.$$

We then have to prove

$$\begin{array}{lll} a1) & g_{a,1}g_{a+1,2}\cdots g_{2a-1,a}g_s = g_{s+a}g_{a,1}g_{a+1,2}\cdots g_{2a-1,a} & \text{for } s\in\{1,2,\ldots,a-1\} \\ b1) & g_{a,1}g_{a+1,2}\cdots g_{2a-1,a}g_s = g_{s-a}g_{a,1}g_{a+1,2}\cdots g_{2a-1,a} & \text{for } s\in\{a+1,a+2,\ldots,2a-1\} \\ c1) & g_{a,1}g_{a+1,2}\cdots g_{2a-1,a}g_s = g_s g_{a,1}g_{a+1,2}\cdots g_{2a-1,a} & \text{otherwise.} \end{array} \tag{3.32}$$

But using only braid relations one checks that $g_{a,b}g_s = g_{s-1}g_{a,b}$ if $s\in\{b+1,\ldots,a\}$, which gives $b1)$. On the other hand, as mentioned above we have that

$$(g_{a,1}g_{a+1,2}\cdots g_{2a-1,a})^* = g_{a,1}g_{a+1,2}\cdots g_{2a-1,a}$$

(actually this can also be shown directly using only the commuting braid relations) and hence the $a1)$ case follows by applying $*$ to the $b1)$ case. The remaining case $c1)$ is easy.

Now the general g_y-case of (3.30) follows from $a), b)$ and $c)$ by expanding $g_y = g_{s_1}\cdots g_{s_l}$ in terms of simple g_s's and pulling $\mathbb{B}_i$ through all factors. Finally the general $\mathbb{B}_w$-case is obtained the same way by expanding $\mathbb{B}_w = \mathbb{B}_{i_1}\cdots\mathbb{B}_{i_l}$ and pulling all factors through.

To show that v is an embedding we argue as in the previous Lemma. Indeed, by construction, the images under v of of the canonical basis vectors of $\mathcal{H}_\alpha^{wr}(q)$ belong to the basis $\mathcal{B}$ for $\mathcal{E}_n^\alpha(q)$ and are pairwise distinct, proving that v is an embedding. □

We are now finally ready to give the construction of the cellular basis for $\mathcal{E}_n^\alpha(q)$. As in the Yokonuma-Hecke algebra case, we first construct, for each $\Lambda\in\mathcal{L}_n(\alpha)$, an element m_Λ that acts as the starting point of the basis. Suppose that $\Lambda = (\boldsymbol{\lambda}\mid\boldsymbol{\mu})$ is as above with $\boldsymbol{\lambda} = (\lambda^{(1)},\ldots,\lambda^{(m)})$ and $\boldsymbol{\mu} = (\mu^{(1)},\ldots,\mu^{(q)})$. We then define m_Λ as follows

$$m_\Lambda := \mathbb{E}_\Lambda x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}}. \tag{3.33}$$

Let us explain the factors of the product. Firstly, $\mathbb{E}_\Lambda$ is the idempotent defined in (3.13). Secondly, $x_{\boldsymbol{\lambda}}\in\mathcal{E}_n^\alpha(q)$ is an analogue for $\mathcal{E}_n^\alpha(q)$ of the element $x_{\boldsymbol{\lambda}}$ for the Hecke algebra, or the element $m_{\boldsymbol{\lambda}}$ in the Yokonuma-Hecke algebra case. It is given as

$$x_{\boldsymbol{\lambda}} := \mathbb{E}_\Lambda \sum_{w\in\mathfrak{S}_{\boldsymbol{\lambda}}} q^{\ell(w)} g_w. \tag{3.34}$$

Mimicking the argument in (6) of Lemma 5 we get that

$$x_{\boldsymbol{\lambda}} g_w = g_w x_{\boldsymbol{\lambda}} = q^{l(w)} x_{\boldsymbol{\lambda}} \quad \text{for } w\in\mathfrak{S}_{\boldsymbol{\lambda}}. \tag{3.35}$$

Finally, in order to explain the factor $b_{\boldsymbol{\mu}}$ we recall from (3.7) the decomposition

$$\mathfrak{S}_\Lambda^m \cong \mathfrak{S}_{m_1}\times\cdots\times\mathfrak{S}_{m_q} \tag{3.36}$$

where $m_i = |\mu^{(i)}|$. Let $x_{\boldsymbol{\mu}}(1)$ be the $q=1$ specialization of the Murphy element corresponding to the multipartition $\boldsymbol{\mu}$, it may be viewed as an element of $S\mathfrak{S}_\Lambda^m$. Then $b_{\boldsymbol{\mu}}$ is defined as

$$b_{\boldsymbol{\mu}} := \iota(x_{\boldsymbol{\mu}}(1)) \in \mathcal{E}_n^\alpha(q) \tag{3.37}$$

where $\iota: S\mathfrak{S}_\Lambda^m \hookrightarrow \mathcal{E}_n^\alpha(q)$ is the embedding from Lemma 19. Let $\mathbb{t}^\Lambda$ be the Λ-tableau given in the obvious way as $\mathbb{t}^\Lambda := (\mathbf{t}^{\boldsymbol{\lambda}}\mid\mathbf{t}^{\boldsymbol{\mu}})$. Then $\mathbb{t}^\Lambda$ is a maximal Λ-tableau, that is the only standard

Λ-tableau $\mathbbm{t}$ satisfying $\mathbbm{t} \trianglerighteq \mathbbm{t}^{\Lambda}$ is $\mathbbm{t}^{\Lambda}$ itself. For $\mathbbm{s} = (\boldsymbol{\mathfrak{s}} \mid \mathbf{u})$ a Λ-tableau we define $d(\mathbbm{s}) := (d(\boldsymbol{\mathfrak{s}}) \mid \iota(d(\mathbf{u}))$ where $d(\boldsymbol{\mathfrak{s}}) \in \mathfrak{S}_n$ as usual is given by $\mathbf{t}^{\boldsymbol{\lambda}} d(\boldsymbol{\mathfrak{s}}) = \boldsymbol{\mathfrak{s}}$ and $d(\mathbf{u}) \in \mathfrak{S}_{\Lambda}^{m}$ by $\mathbf{t}^{\boldsymbol{\mu}} d(\mathbf{u}) = \mathbf{u}$. For simplicity, we often write $(d(\boldsymbol{\mathfrak{s}}) \mid d(\mathbf{u}))$ for $(d(\boldsymbol{\mathfrak{s}}) \mid \iota(d(\mathbf{u})))$. Note that since $\mathbf{u} = (\mathfrak{u}_1, \dots, \mathfrak{u}_q)$ is always of the initial kind, we have a decomposition $d(\mathbf{u}) = (d(\mathfrak{u}_1), \dots, d(\mathfrak{u}_q))$, according to (3.36), and also

$$\mathbb{B}_{d(\mathbf{u})} = \mathbb{B}_{d(\mathfrak{u}_1)} \cdots \mathbb{B}_{d(\mathfrak{u}_q)}.$$

Finally, we define the main object of this section. For $\mathbbm{s} = (\boldsymbol{\mathfrak{s}} \mid \mathbf{u})$, $\mathbbm{t} = (\boldsymbol{\mathfrak{t}} \mid \mathbf{v})$ row standard Λ-tableaux we define

$$m_{\mathbbm{s}\mathbbm{t}} := g^{*}_{d(\boldsymbol{\mathfrak{s}})} \mathbb{E}_{\Lambda} \mathbb{B}^{*}_{d(\mathbf{u})} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, g_{d(\boldsymbol{\mathfrak{t}})}. \tag{3.38}$$

Our aim is to prove that the $m_{\mathbbm{s}\mathbbm{t}}$'s, with $\mathbbm{s}$ and $\mathbbm{t}$ running over standard Λ-tableaux, form a cellular basis for $\mathcal{E}_n^{\alpha}(q)$. To achieve this goal we first need to work out commutation rules between the various ingredients of $m_{\mathbbm{s}\mathbbm{t}}$. The rules shall be formulated in terms of a certain $\circ$-action on tableaux that we explain now.

Let $B_y \in \mathfrak{S}_{\Lambda}^{k}$. From now on, when confusion should not be possible, we shall write $\boldsymbol{\mathfrak{s}} y$ for $\boldsymbol{\mathfrak{s}} B_y$ where $\boldsymbol{\mathfrak{s}}$ is the first part of a Λ-tableau and where the action of B_y is given by the complete expansion of B_y in terms of s_i's.

Let $\mathbbm{s} = (\boldsymbol{\mathfrak{s}} \mid \mathbf{u})$ be a Λ-tableau. We then define a new multitableau $y \circ \boldsymbol{\mathfrak{s}}$ as follows. Set first $\boldsymbol{\mathfrak{s}}_1 := \boldsymbol{\mathfrak{s}} y^{-1} = (\mathfrak{s}_1^{(1)}, \dots, \mathfrak{s}_1^{(m)})$. Then $y \circ \boldsymbol{\mathfrak{s}}$ is given by the formula

$$y \circ \boldsymbol{\mathfrak{s}} := (\mathfrak{s}_1^{(1)y}, \dots, \mathfrak{s}_1^{(m)y}). \tag{3.39}$$

With this notation we have the following Lemma which is easy to verify.

LEMMA 21. *The map $(y, \boldsymbol{\mathfrak{s}}) \mapsto y \circ \boldsymbol{\mathfrak{s}}$ defines a left action of $\mathfrak{S}_{\Lambda}^{k}$ on the set of multitableaux $\boldsymbol{\mathfrak{s}}$ such that $Shape(\boldsymbol{\mathfrak{s}}) =_1 \boldsymbol{\lambda}$ where $\boldsymbol{\lambda}$ is the first part of a Λ-tableau; that is $Shape(\boldsymbol{\mathfrak{s}})$ and $\boldsymbol{\lambda}$ are equal multipartitions up to a permutation. Moreover, if $\boldsymbol{\mathfrak{s}}$ is of the initial kind then also $y \circ \boldsymbol{\mathfrak{s}}$ is of the initial kind, and if $y \in \mathfrak{S}_{\Lambda}^{m}$ then $y \circ \boldsymbol{\mathfrak{s}} = \boldsymbol{\mathfrak{s}}$.*

EXAMPLE 5. *We give an example to illustrate the action. As can be seen, it permutes the partitions of the multitableau, but keeps the numbers. Consider*

$$\boldsymbol{\mathfrak{s}} := \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array}, \begin{array}{|c|c|}\hline 7&9\\\hline 8\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 10&11&12\\\hline\end{array}, \begin{array}{|c|}\hline 13\\\hline 14\\\hline 15\\\hline\end{array} \right) \quad \textit{and} \quad B_y := B_1 B_2 B_1 B_4 B_5.$$

We first note that $\boldsymbol{\mathfrak{s}} y^{-1} = \left(\begin{array}{|c|c|}\hline 5&6\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 10&12\\\hline 11\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 13&14&15\\\hline\end{array}, \begin{array}{|c|}\hline 7\\\hline 8\\\hline 9\\\hline\end{array} \right)$. *Then we have*

$$y \circ \boldsymbol{\mathfrak{s}} = \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}, \begin{array}{|c|c|}\hline 5&6\\\hline\end{array}, \begin{array}{|c|}\hline 7\\\hline 8\\\hline 9\\\hline\end{array}, \begin{array}{|c|c|}\hline 10&12\\\hline 11\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 13&14&15\\\hline\end{array} \right).$$

Let $\boldsymbol{\mathfrak{s}}$ and $\boldsymbol{\mathfrak{t}}$ be $\boldsymbol{\lambda}$-multitableaux. Then we define $x_{\boldsymbol{\mathfrak{s}}\boldsymbol{\mathfrak{t}}} \in \mathcal{E}_n^{\alpha}(q)$, just as for the Yokonuma-Hecke algebra, that is

$$x_{\boldsymbol{\mathfrak{s}}\boldsymbol{\mathfrak{t}}} := g^{*}_{d(\boldsymbol{\mathfrak{s}})} x_{\boldsymbol{\lambda}} g_{d(\boldsymbol{\mathfrak{t}})} \in \mathcal{E}_n^{\alpha}(q). \tag{3.40}$$

The following remark is an analogue of Remark 5 for the Yokonuma-Hecke algebra.

REMARK 11. *Let* $\mathfrak{s}$ *and* $\mathfrak{t}$ *be multitableaux of the initial kind and let*

$$d(\mathfrak{s}) = \Big(d(\mathfrak{s}^{(1)}), d(\mathfrak{s}^{(2)}), \cdots, d(\mathfrak{s}^{(m)})\Big) \quad \textit{and} \quad d(\mathfrak{t}) = \Big(d(\mathfrak{t}^{(1)}), d(\mathfrak{t}^{(2)}), \cdots, d(\mathfrak{t}^{(m)})\Big)$$

be the decompositions given in (3.16). Then, under the embedding from (3.27) of the Young-Hecke algebra

$$\epsilon : \mathcal{H}_\alpha(q) \hookrightarrow \mathcal{E}_n^\alpha(q)$$

we have that $\epsilon(x_{\mathfrak{s}^{(1)}\mathfrak{t}^{(1)}} \otimes x_{\mathfrak{s}^{(2)}\mathfrak{t}^{(2)}} \otimes \cdots \otimes x_{\mathfrak{s}^{(m)}\mathfrak{t}^{(m)}}) = x_{\mathfrak{s}\mathfrak{t}}$.

The next Lemma gives the promised commutation formulas.

LEMMA 22. *Suppose* $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ *and* $\mathbb{t} = (\mathfrak{t} \mid \mathbf{v})$ *are* Λ*-tableaux such that* $\mathfrak{s}$ *and* $\mathfrak{t}$ *are of the initial kind and suppose that* $B_y \in \mathfrak{S}_\Lambda^k$. *Then we have the following formulas in* $\mathcal{E}_n^\alpha(q)$.

(1) $\mathbb{E}_\Lambda \mathbb{B}_y g_{d(\mathfrak{s})} = \mathbb{E}_\Lambda g_{d(y\circ\mathfrak{s})} \mathbb{B}_y$.
(2) $\mathbb{E}_\Lambda \mathbb{B}_y x_{\mathfrak{s}\mathfrak{t}} = \mathbb{E}_\Lambda x_{y\circ\mathfrak{s}, y\circ\mathfrak{t}} \mathbb{B}_y$.

PROOF. In order to prove (1) we may assume that $\mathbb{B}_y = \mathbb{B}_i$, since $y \mapsto y \circ \mathfrak{s}$ is a left action. Now $\mathfrak{s} = (\mathfrak{s}^{(1)}, \ldots, \mathfrak{s}^{(m)})$ is of the initial kind and so we have a decomposition $g_{d(\mathfrak{s})} = g_{d(\mathfrak{s}^{(1)})} \cdots g_{d(\mathfrak{s}^{(m)})}$ with the $g_{d(\mathfrak{s}^{(i)})}$'s belonging to Hecke algebras running over the distinct indices given by the symmetric group factors $\mathfrak{S}_{k_i}$ of $\mathfrak{S}_\Lambda^k$. Then by (3.31) we have that

$$\mathbb{E}_\Lambda \mathbb{B}_i g_{d(\mathfrak{s})} = \mathbb{E}_\Lambda \mathbb{B}_i g_{d(\mathfrak{s}^{(1)})} \cdots g_{d(\mathfrak{s}^{(m)})} = \mathbb{E}_\Lambda g_{d(\mathfrak{r}^{(1)})} \cdots g_{d(\mathfrak{r}^{(m)})} \mathbb{B}_i$$

where $d(\mathfrak{s}^{(k)}) = d(\mathfrak{r}^{(k)})$ for $k \neq i, i+1$ and where $d(\mathfrak{s}^{(i)}), d(\mathfrak{r}^{(i+1)}), d(\mathfrak{s}^{(i+1)}), d(\mathfrak{r}^{(i)})$ are related as in (3.31): each factor g_s of $d(\mathfrak{s}^{(i)})$ is replaced by g_{s+k_i} to arrive at $d(\mathfrak{r}^{(i+1)})$ and similarly for $d(\mathfrak{s}^{(i+1)})$ and $d(\mathfrak{r}^{(i)})$. But this means exactly that

$$g_{d(B_i\circ\mathfrak{s})} = g_{d(\mathfrak{r}^{(1)})} \cdots g_{d(\mathfrak{r}^{(m)})}$$

and so (1) follows.

On the other hand, applying $*$ to (1) and using that $\mathbb{B}_y^* = \mathbb{B}_{y^{-1}}$, we find $\mathbb{E}_\Lambda \mathbb{B}_{y^{-1}} g^*_{d(y\circ\mathfrak{s})} = \mathbb{E}_\Lambda g^*_{d(\mathfrak{s})} \mathbb{B}_{y^{-1}}$, that is

$$\mathbb{E}_\Lambda \mathbb{B}_y g^*_{d(\mathfrak{s})} = \mathbb{E}_\Lambda g^*_{d(y\circ\mathfrak{s})} \mathbb{B}_y.$$

(Alternatively, one can also repeat the argument for (1)). Now (3.30) can be formulated as follows

$$\mathbb{E}_\Lambda \mathbb{B}_y g_k = \mathbb{E}_\Lambda g_{kB_y^{-1}} \mathbb{B}_y \tag{3.41}$$

and hence we get $\mathbb{E}_\Lambda \mathbb{B}_y x_{\boldsymbol{\lambda}} = \mathbb{E}_\Lambda x_{\boldsymbol{\mu}} \mathbb{B}_y$, where $\boldsymbol{\mu} = Shape(y \circ \mathfrak{t}^{\boldsymbol{\lambda}})$. In view of the definitions this shows (2). □

COROLLARY 3.2. *The factor* $x_{\boldsymbol{\lambda}}$ *of* $m_{\mathbb{s}\mathbb{t}}$ *commutes with each of the factors* $\mathbb{B}^*_{d(\mathbf{u})}$, $b_{\boldsymbol{\mu}}$ *and* $\mathbb{B}_{d(\mathbf{v})}$ *of* $m_{\mathbb{s}\mathbb{t}}$. *Furthermore,*

$$m^*_{\mathbb{s}\mathbb{t}} = m_{\mathbb{t}\mathbb{s}}. \tag{3.42}$$

PROOF. Setting $\mathfrak{s} = \mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$ in part (2) of the Lemma we get for $\mathbb{B}_y \in \mathfrak{S}_\Lambda^m$ that

$$\mathbb{E}_\Lambda \mathbb{B}_y x_{\boldsymbol{\lambda}} = \mathbb{E}_\Lambda x_{y\circ \mathfrak{t}^{\boldsymbol{\lambda}}, y\circ \mathfrak{t}^{\boldsymbol{\lambda}}} \mathbb{B}_y = \mathbb{E}_\Lambda x_{\boldsymbol{\lambda}} \mathbb{B}_y \tag{3.43}$$

since $\mathfrak{t}^{\boldsymbol{\lambda}}$ is of the initial kind and therefore $y \circ \mathfrak{t}^{\boldsymbol{\lambda}} = \mathfrak{t}^{\boldsymbol{\lambda}}$ by Lemma 21. This shows the first claim. To show the second claim, we use the first claim together with $\mathbb{E}_\Lambda \mathbb{B}_y = \mathbb{B}_y \mathbb{E}_\Lambda$ for all $y \in \mathfrak{S}_\Lambda^k$, as follows from Proposition 3.1(1) and the definition of $\mathbb{B}_i$, to get

$$m_{\mathfrak{s}\mathfrak{t}}^* = g_{d(\mathfrak{t})}^* \mathbb{B}_{d(\mathbf{v})}^* b_{\boldsymbol{\mu}}^* x_{\boldsymbol{\lambda}}^* \mathbb{B}_{d(\mathbf{u})} \mathbb{E}_\Lambda^* g_{d(\mathfrak{s})} = g_{d(\mathfrak{t})}^* \mathbb{E}_\Lambda \mathbb{B}_{d(\mathbf{v})}^* x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{u})} g_{d(\mathfrak{s})} = m_{\mathfrak{t}\mathfrak{s}}$$

as claimed. □

We need the following technical Lemma.

LEMMA 23. *Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ such that $\mathfrak{s}$ is a $\boldsymbol{\lambda}$-multitableau. Let $w_{\mathfrak{s}}$ be the distinguished representative for $d(\mathfrak{s})$ with respect to $\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$, that is we have the decomposition $d(\mathfrak{s}) = d(\mathfrak{s}_0) w_{\mathfrak{s}}$, as in (2.3). Let $B_y \in \mathfrak{S}_\Lambda^k$. Then, in $\mathcal{E}_n^\alpha(q)$ we have the identity $\mathbb{B}_y g_{w_{\mathfrak{s}}} = \mathbb{E}_\Lambda g_{B_y w_{\mathfrak{s}}}$ (even though in general $l(B_y w_{\mathfrak{s}}) \neq l(B_y) + l(w_{\mathfrak{s}})$). Moreover, for any multitableau $\mathfrak{t}_0$ of the initial kind with respect to $\boldsymbol{\lambda}$, we have that $\mathbb{E}_\Lambda g_{d(\mathfrak{t}_0) B_y w_{\mathfrak{s}}} = \mathbb{E}_\Lambda g_{d(\mathfrak{t}_0)} \mathbb{B}_y g_{w_{\mathfrak{s}}}$.*

PROOF. The ingredients of the proof are already present in the proof of part b) of Lemma 19. As before we set $A_{\boldsymbol{\lambda}} = \{I_1, I_2, \dots, I_q\}$, with blocks I_i. Let $B_y = s_{i_1} \dots s_{i_r}$ be the expansion of B_y according to the definitions and let $w_{\mathfrak{s}} = s_{j_1} \dots s_{j_s}$ be a reduced expression. The action of B_y involves at each step distinct blocks, that is i_k and $i_k + 1$ occur in distinct blocks of $(A_{\boldsymbol{\lambda}}) s_{i_1} \dots s_{i_{k-1}}$ for all k. A similar property holds for $w_{\mathfrak{s}}$ since it is the distinguished coset representative for $d(\mathfrak{s})$ with respect to $\mathfrak{S}_{\boldsymbol{\lambda}}$. But the blocks of $(A_{\boldsymbol{\lambda}}) B_y$ are a permutation of the blocks of $A_{\boldsymbol{\lambda}}$, that is $(A_{\boldsymbol{\lambda}}) B_y = A_{\boldsymbol{\lambda}}$ as set partitions, and so also the action of the concatenation $s_{i_1} \dots s_{i_r} s_{j_1} \dots s_{j_s}$ on $A_{\boldsymbol{\lambda}}$ involves at each step distinct blocks.

We now transform $s_{i_1} \dots s_{i_r} s_{j_1} \dots s_{j_s}$ into a reduced expression for $B_y w_{\mathfrak{s}}$ using the Coxeter relations of type A. We claim that these Coxeter relations map a sequence $s_{\iota_1} \dots s_{\iota_t}$ having the property of acting at each step in distinct blocks to another sequence having the same property. This is clear for the commuting Coxeter relations $s_i s_j = s_j s_i$ and also for the quadratic relations $s_i^2 = 1$. In the case of the braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ we observe that both $s_i s_{i+1} s_i$ and $s_{i+1} s_i s_{i+1}$ have the above property with respect to $A = \{J_1, \dots, J_u\}$ exactly when all three numbers $i, i+1$ and $i+2$ occur in three distinct blocks J_i of A and so the claim follows also in that case.

Now by definition $\mathbb{B}_y g_{w_{\mathfrak{s}}} = \mathbb{E}_\Lambda g_{i_1} \dots g_{i_r} g_{j_1} \dots g_{j_s}$ and so the above sequence of Coxteter relations will transform $\mathbb{B}_y g_{w_{\mathfrak{s}}}$ to $\mathbb{E}_\Lambda g_{B_y w_{\mathfrak{s}}}$. Indeed, for each occurence of the relation $s_i^2 = 1$ we have by part (3) of Proposition 3.1 a corresponding relation

$$\mathbb{E}_A g_i^2 = \mathbb{E}_A (1 + (q - q^{-1}) e_i g_i) = \mathbb{E}_A \tag{3.44}$$

whenever i and $i+1$ are in distinct blocks of A. This proves the first statement of the Lemma. The second statement follows from the first since $B_y w_{\mathfrak{s}}$ is the distinguished representative for its class with respect to $\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ as follows from the characterization of distinguished representatives as row standard tableaux, of shape $\beta = \|\boldsymbol{\lambda}\|^{op}$ in this case. Indeed, $\mathfrak{t}^\beta B_y w_{\mathfrak{s}}$ is obtained from $\mathfrak{t}^{\beta} w_{\mathfrak{s}}$ by permuting some rows and so one is row standard iff the other is row standard. $\square$

The following Lemma is the $\mathcal{E}_n^{\alpha}(q)$-version of Lemma 9 in the Yokonuma-Hecke algebra case.

LEMMA 24. *Suppose that $\Lambda \in \mathcal{L}_n(\alpha)$ and that $\mathbb{s} = (\mathfrak{s} \,|\, \mathbf{u})$ and $\mathbb{t} = (\mathbf{t} \,|\, \mathbf{v})$ are row standard Λ-tableaux. Then for every $h \in \mathcal{E}_n^{\alpha}(q)$ we have that $m_{\mathbb{s}\mathbb{t}} h$ is a linear combination of terms of the form $m_{\mathbb{s}\mathbb{v}}$ where $\mathbb{v}$ is a row standard Λ-tableau. A similar statement holds for $h m_{\mathbb{s}\mathbb{t}}$.*

PROOF. The idea is to repeat the arguments of Lemma 9. It is enough to consider the $m_{\mathbb{s}\mathbb{t}} h$ case. Using the Corollary we have that

$$m_{\mathbb{s}\mathbb{t}} = g^*_{d(\mathfrak{s})} \mathbb{E}_{\Lambda} \mathbb{B}^*_{d(\mathbf{u})} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, g_{d(\mathbf{t})} = g^*_{d(\mathfrak{s})} \mathbb{B}^*_{d(\mathbf{u})} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})} x_{\boldsymbol{\lambda}} \mathbb{E}_{\Lambda}\, g_{d(\mathbf{t})}. \tag{3.45}$$

Since h is general we reduce to the case $g_{d(\mathbf{t})} = 1$, that is $\mathbf{t} = \mathbf{t}^{\boldsymbol{\lambda}}$. We may assume that $h = \mathbb{E}_A g_v$ since such elements form a basis for $\mathcal{E}_n^{\alpha}(q)$. But the $\mathbb{E}_A$'s are orthogonal idempotents, as was shown in Proposition 3.1, and so we may further reduce to the case $h = g_v$. We have a decomposition $v = v_0 d(\mathfrak{v})$ with $v_0 \in \mathfrak{S}_{\boldsymbol{\lambda}}$ and $\mathfrak{v}$ a row standard $\boldsymbol{\lambda}$-multitableau such that $l(w) = l(v_0) + l(d(\mathfrak{v}))$. Hence via (3.35) we get that $m_{\mathbb{s}\mathbb{t}}\, g_v$ is a multiple of

$$g^*_{d(\mathfrak{s})} \mathbb{B}^*_{d(\mathbf{u})} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})} \mathbb{E}_{\Lambda} x_{\boldsymbol{\lambda}}\, g_{d(\mathfrak{v})} = g^*_{d(\mathfrak{s})} \mathbb{B}^*_{d(\mathbf{u})} \mathbb{E}_{\Lambda} b_{\boldsymbol{\mu}} x_{\boldsymbol{\lambda}} \mathbb{B}_{d(\mathbf{v})}\, g_{d(\mathfrak{v})} = m_{\mathbb{s}\mathbb{v}} \tag{3.46}$$

where $\mathbb{v} = (\mathfrak{v} \,|\, \mathbf{v})$. $\square$

Our next Lemma is the analogue for $\mathcal{E}_n^{\alpha}(q)$ of Lemma 10. It is the key Lemma for our results on $\mathcal{E}_n^{\alpha}(q)$.

LEMMA 25. *Suppose that $\Lambda \in \mathcal{L}_n(\alpha)$ and that $\mathbb{s}$ and $\mathbb{t}$ are row standard Λ-tableaux. Then there are standard tableaux $\mathbb{u}$ and $\mathbb{v}$ such that $\mathbb{u} \trianglerighteq \mathbb{s}, \mathbb{v} \trianglerighteq \mathbb{t}$ and such that $m_{\mathbb{s}\mathbb{t}}$ is a linear combination of the elements $m_{\mathbb{u}\mathbb{v}}$.*

PROOF. Let $\Lambda = (\boldsymbol{\lambda} \,|\, \boldsymbol{\mu})$, $\mathbb{s} = (\mathfrak{s} \,|\, \mathbf{u})$ and $\mathbb{t} = (\mathbf{t} \,|\, \mathbf{v})$. Then we have

$$m_{\mathbb{s}\mathbb{t}} = g^*_{d(\mathfrak{s})} \mathbb{E}_{\Lambda} \mathbb{B}^*_{d(\mathbf{u})} b_{\boldsymbol{\mu}} x_{\boldsymbol{\lambda}} \mathbb{B}_{d(\mathbf{v})}\, g_{d(\mathbf{t})}. \tag{3.47}$$

Suppose first that standardness fails for $\mathfrak{s}$ or $\mathbf{t}$. The basic idea is then to proceed as in the proof of Lemma 10. There exist multitableaux $\mathfrak{s}_0$ and $\mathbf{t}_0$ of the initial kind together with $w_{\mathfrak{s}}, w_{\mathbf{t}} \in \mathfrak{S}_n$ such that $d(\mathfrak{s}) = d(\mathfrak{s}_0) w_{\mathfrak{s}}$, $d(\mathbf{t}) = d(\mathbf{t}_0) w_{\mathbf{t}}$ and $\ell(d(\mathfrak{s})) = \ell(d(\mathfrak{s}_0)) + \ell(w_{\mathfrak{s}})$ and $\ell(\mathbf{t})) = \ell(d(\mathbf{t}_0)) + \ell(w_{\mathbf{t}})$. That is, $w_{\mathfrak{s}}$ and $w_{\mathbf{t}}$ are distinguished right coset representatives for $d(\mathfrak{s})$ and $d(\mathbf{t})$ with respect to $\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ and (3.47) becomes

$$m_{\mathbb{s}\mathbb{t}} = g^*_{w_{\mathfrak{s}}} g^*_{d(\mathfrak{s}_0)} \mathbb{E}_{\Lambda} \mathbb{B}^*_{d(\mathbf{u})} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}\, g_{d(\mathbf{t}_0)}\, g_{w_{\mathbf{t}}} \tag{3.48}$$

since the two middle terms commute. Note that the factor $\mathbb{E}_{\Lambda}$ commutes with all other except the two extremal factors of (3.48). Expanding $b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})}$ completely as a linear combination of $\mathbb{B}_{y_{\mathbf{t}}}$'s with $B_{y_{\mathbf{t}}} \in \mathfrak{S}^m_{\Lambda}$ and setting $\mathbb{B}_{y_{\mathfrak{s}}} := \mathbb{B}_{d(\mathbf{u})}$ we get via Lemma 21 and Lemma 22 that (3.48) is a linear combination of terms

$$g^*_{w_{\mathfrak{s}}} \mathbb{B}^*_{y_{\mathfrak{s}}} x_{\mathfrak{s}_0 \mathbf{t}_0} \mathbb{B}_{y_{\mathbf{t}}}\, g_{w_{\mathbf{t}}} = g^*_{w_{\mathfrak{s}}} x_{\mathfrak{s}_0 \mathbf{t}_0} \mathbb{B}_{y}\, g_{w_{\mathbf{t}}} \tag{3.49}$$

where $\mathbb{B}_y = \mathbb{B}^*_{y_{\mathfrak{s}}}\mathbb{B}_{y_{\mathfrak{t}}}$. For each appearing $B_y \in \mathfrak{S}^m_\Lambda$ we have by Lemma 23 that

$$\mathbb{E}_\Lambda g_{B_y} g_{w_{\mathfrak{t}}} = \mathbb{E}_\Lambda g_{B_y w_{\mathfrak{t}}}.$$

Thus (3.49) becomes a linear combination of terms

$$g^*_{w_{\mathfrak{s}}}\mathbb{E}_\Lambda x_{\mathfrak{s}_0\mathfrak{t}_0} g_{y_{\mathfrak{t},1}} \tag{3.50}$$

where $y_{\mathfrak{t},1} := B_y w_{\mathfrak{t}}$. We now proceed as in the Yokonuma-Hecke algebra case. Via Remark 11 we apply Murphy's result [**37**, Theorem 4.18] on $x_{\mathfrak{s}_0\mathfrak{t}_0}$, thus rewriting it as a linear combination of $x_{\mathfrak{s}_1\mathfrak{t}_1}$ where $\mathfrak{s}_1$ and $\mathfrak{t}_1$ are standard $\boldsymbol{\nu}$-multitableaux of the initial kind, satisfying $\mathfrak{s}_1 \trianglerighteq \mathfrak{s}_0$ and $\mathfrak{t}_1 \trianglerighteq \mathfrak{t}_0$. We then get that (3.49) is a linear combination of such terms

$$g^*_{w_{\mathfrak{s}}}\mathbb{E}_\Lambda x_{\mathfrak{s}_1\mathfrak{t}_1} g_{y_{\mathfrak{t},1}}. \tag{3.51}$$

Let $\boldsymbol{\nu} = (\nu^{(1)},\dots,\nu^{(m)})$. It need not be an increasing multipartition and our task is to fix this problem.

We determine a $B_\sigma \in \mathfrak{S}^k_\Lambda$ such that the multipartition $\boldsymbol{\nu}^{ord} := (\nu^{(1)\sigma},\dots,\nu^{(m)\sigma})$ is increasing. Then, using (2) of Lemma 22 we get that (3.51) is equal to

$$g^*_{w_{\mathfrak{s}}}\mathbb{B}^*_\sigma\mathbb{B}_\sigma x_{\mathfrak{s}_1\mathfrak{t}_1} g_{y_{\mathfrak{t},1}} = g^*_{w_{\mathfrak{s}}}\mathbb{B}^*_\sigma x_{\sigma\circ\mathfrak{s}_1,\sigma\circ\mathfrak{t}_1}\mathbb{B}_\sigma g_{y_{\mathfrak{t},1}} = g^*_{y_{\mathfrak{s},2}} x_{\sigma\circ\mathfrak{s}_1,\sigma\circ\mathfrak{t}_1} g_{y_{\mathfrak{t},2}} \tag{3.52}$$

where $y_{\mathfrak{s},2} := B_\sigma w_{\mathfrak{s}}$ and $y_{\mathfrak{t},2} := B_\sigma y_{\mathfrak{t},1}$, and where we used Lemma 23 once again. Here $\mathfrak{t}^{\boldsymbol{\nu}^{ord}} y_{\mathfrak{s},2}$ and $\mathfrak{t}^{\boldsymbol{\nu}^{ord}} y_{\mathfrak{t},2}$ are standard $\boldsymbol{\nu}^{ord}$-multitableaux but not necessarily increasing, and so we must now fix this problem. Let therefore $\mathfrak{S}^{m'}$ be the subgroup of $\mathfrak{S}^k_\Lambda$ that permutes equal $\nu^{(i)}$'s. We can then find $\sigma_1,\sigma_2 \in \mathfrak{S}^{m'}_\Lambda$ such that $\mathfrak{t}^{\boldsymbol{\nu}^{ord}} B_{\sigma_1} y_{\mathfrak{s},2}$ and $\mathfrak{t}^{\boldsymbol{\nu}^{ord}} B_{\sigma_2} y_{\mathfrak{t},2}$ are increasing $\boldsymbol{\nu}^{ord}$-tableaux. With these choices, (3.52) becomes via Lemma 23

$$g^*_{y_{\mathfrak{s},3}} x_{\sigma\circ\mathfrak{s}_1,\sigma\circ\mathfrak{t}_1}\mathbb{B}_{\sigma_1}\mathbb{B}^*_{\sigma_2} g_{y_{\mathfrak{t},3}} \tag{3.53}$$

where $y_{\mathfrak{s},3} := B_{\sigma_1} y_{\mathfrak{s},2}$ and $y_{\mathfrak{t},3} := B_{\sigma_2} y_{\mathfrak{t},2}$, and where we used (2) of Lemma 22 to show that $\sigma\circ\mathfrak{s}_1$ and $\sigma\circ\mathfrak{t}_1$ are unchanged by the commutation with $\mathbb{B}_{\sigma_1}$. We now set $\mathfrak{s}_3 := \mathfrak{t}^{\boldsymbol{\nu}^{ord}} d(\sigma\circ\mathfrak{s}_1) y_{\mathfrak{s},3}$ where $d(\sigma\circ\mathfrak{s}_1)$ is calculated with respect to $Shape(\sigma\circ\mathfrak{s}_1) = \mathfrak{t}^{\boldsymbol{\nu}^{ord}}$ of course, and similarly $\mathfrak{t}_3 := \mathfrak{t}^{\boldsymbol{\nu}^{ord}} d(\sigma\circ\mathfrak{t}_1) y_{\mathfrak{t},3}$. Then $\mathfrak{s}_3$ and $\mathfrak{t}_3$ are increasing standard multitableaux of shape $\boldsymbol{\nu}^{ord}$ and we get via Lemma 23 that (3.53) is equal to

$$g^*_{d(\mathfrak{s}_3)} x_{\boldsymbol{\nu}^{ord}}\mathbb{B}_{\sigma_1}\mathbb{B}^*_{\sigma_2} g_{d(\mathfrak{t}_3)} \tag{3.54}$$

since $g_{d(\sigma\circ\mathfrak{t}_1)}$ and $\mathbb{B}^*_{\sigma_1}\mathbb{B}_{\sigma_2}$ commute by Lemma 22.

In order to show that (3.47) has the form $m_{\mathbb{u}\mathbb{v}}$ stipulated by the Lemma, we must now treat the factor $\mathbb{B}_{\sigma_1}\mathbb{B}^*_{\sigma_2}$. But since $\mathfrak{S}^{m'}$ is a product of symmetric groups, $B_{\sigma_1}B^*_{\sigma_2}$ can be written as a linear combination of $x_{\mathbf{u}'\mathbf{v}'}(1)$, with $\mathbf{u}'$ and $\mathbf{v}'$ running over multitableaux of the initial kind according to the factors of $\mathfrak{S}^{m'}$ and where once again $x_{\mathbf{u}'\mathbf{v}'}(1)$ is the usual Murphy standard basis element, evaluated at 1, for that product. Thus (3.54) becomes

$$g^*_{d(\mathfrak{s}_3)}\mathbb{B}^*_{\mathbf{u}'} b_{\boldsymbol{\mu}} x_{\boldsymbol{\nu}^{ord}}\mathbb{B}_{\mathbf{v}'} g_{d(\mathfrak{t}_3)} = m_{\mathbb{u}\mathbb{v}} \tag{3.55}$$

where $\mathbb{u} = (\mathfrak{s}_3 \mid \mathbf{u}')$ and $\mathbb{v} = (\mathfrak{t}_3 \mid \mathbf{v}')$. Note that by the constructions we have that the shape of $\mathfrak{s}_3$ and $\mathfrak{t}_3$ is of type α and that $\mathbb{u} \trianglerighteq \mathbb{s}, \mathbb{v} \trianglerighteq \mathbb{t}$ and so the Lemma is proved in this case.

Finally, we must now treat the case where standardness holds for $\mathfrak{s}$ and $\mathfrak{t}$, but fails for $\mathbf{u}$ or $\mathbf{v}$. But this case is much easier, since we can here apply Murphy's theory directly, thus expanding the nonstandard terms in terms of standard terms.

□

We are now ready to state and prove the main Theorem of this section.

THEOREM 3.2. *Let* $\mathcal{BT}_n := \{m_{\mathbb{s}\mathbb{t}} \,|\, \mathbb{s},\mathbb{t} \in \mathrm{Std}(\Lambda), \Lambda \in \mathcal{L}_n\}$ *and* $\mathcal{BT}_n^{\alpha} := \{m_{\mathbb{s}\mathbb{t}} \,|\, \mathbb{s},\mathbb{t} \in \mathrm{Std}(\Lambda), \Lambda \in \mathcal{L}_n(\alpha)\}$ *for* $\alpha \in \mathcal{P}ar_n$. *Then* $\mathcal{BT}$ *is a cellular basis for* $\mathcal{E}_n(q)$ *and* $\mathcal{BT}_n^{\alpha}$ *is a cellular basis for* $\mathcal{E}_n^{\alpha}(q)$.

PROOF. By the decomposition in (2.4) it is enough to show that $\mathcal{BT}_n^{\alpha}$ is a cellular basis for $\mathcal{E}_n^{\alpha}(q)$. Let $\mathbb{E}_\Lambda$ be the idempotent corresponding to any element of $\Lambda \in \mathcal{L}_n(\alpha)$: in fact $\mathbb{E}_\Lambda$ is independent of the choice of $\Lambda \in \mathcal{L}_n(\alpha)$. Then the set $\{g_w \mathbb{E}_\Lambda g_{w^1} \,|\, w, w^1 \in \mathfrak{S}_n\}$ generates $\mathcal{E}_n^{\alpha}(q)$ over S. Thus letting $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$ vary over pairs of one-column multipartitions with $\boldsymbol{\lambda}$ of type α and letting $\mathbb{s},\mathbb{t}$ vary over row standard Λ-tableaux, we get that the corresponding $m_{\mathbb{s}\mathbb{t}}$ generate $\mathcal{E}_n^{\alpha}(q)$ over S. Indeed, for such Λ we have that $\boldsymbol{\lambda}$ is a one-column multipartition and therefore $\mathfrak{t}^{\boldsymbol{\lambda}} w$ is row standard for any w. Moreover, for such Λ the row stabilizer of $\boldsymbol{\mu}$ is trivial and therefore $b_{\boldsymbol{\mu}}$ is just the identity element of $\mathfrak{S}_\Lambda^m$. In other words, any $g_w \mathbb{E}_\Lambda g_{w^1}$ can be realized in the form $m_{\mathbb{s}\mathbb{t}}$ for Λ-tableaux $\mathbb{s}$ and $\mathbb{t}$.

But then, using the last two Lemmas, we deduce that the elements from $\mathcal{BT}_n^{\alpha}$ generate $\mathcal{E}_n^{\alpha}(q)$ over S. On the other hand, by the proof of Lemma 18 these elements have cardinality equal to $\dim \mathcal{E}_n^{\alpha}(q)$, and so they indeed form a basis for $\mathcal{E}_n^{\alpha}(q)$, as can be seen by repeating the argument of Theorem 2.6.

The $*$-condition for cellularity has already been checked above in (3.42). Finally, to show the multiplication condition for $\mathcal{BT}_n^{\alpha}$ to be cellular, we can repeat the argument from the Yokonuma-Hecke algebra case. Indeed, to $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$ we have associated the Λ-tableau $\mathbb{t}^{\Lambda}$ and have noticed that the only standard Λ-tableau $\mathbb{t}$ satisfying $\mathbb{t} \trianglerighteq \mathbb{t}^{\Lambda}$ is $\mathbb{t}^{\Lambda}$ itself. The Theorem follows from this just like in the Yokonuma-Hecke algebra case. □

COROLLARY 3.3. *The dimension of the cell module* $C(\Lambda)$ *associated with* $\Lambda \in \mathcal{L}_n$ *is given by the formula of Corollary 3.1.*

COROLLARY 3.4. *Let* α *be a partition of* n. *Recall the set* $\mathcal{L}_n(\alpha)$ *introduced in the proof of Lemma 18. Then* $\mathcal{BT}_n^{\alpha} := \{m_{\mathbb{s}\mathbb{t}} \,|\, \mathbb{s},\mathbb{t} \in \mathrm{Std}(\Lambda), \Lambda \in \mathcal{L}_n(\alpha)\}$ *is a cellular basis for* $\mathcal{E}_n^{\alpha}(q)$.

Unlike $\mathfrak{S}_\Lambda^m$, the group $\mathfrak{S}_\Lambda^k$ has so far not played any important role in the article, but now it enters the game. We need the following definition.

DEFINITION 3.2. *Let* $\Lambda \in \mathcal{L}_n(\alpha)$ *for* $\alpha \in \mathcal{P}ar_n$ *and let* $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ *be a* Λ*-tableau. Then we say that* $\mathbb{s}$ *(and* $\mathfrak{s}$*) is of wreath type for* Λ *if* $\mathfrak{s} = \mathfrak{s}_0 y$ *for some* $B_y \in \mathfrak{S}_\Lambda^k$ *where* $\mathfrak{s}_0$ *a multitableau of the initial kind. Moreover we define*

$$\mathcal{BT}_n^{\alpha,wr} := \{m_{\mathbb{s}\mathbb{t}} \,|\, \mathbb{s},\mathbb{t} \in \mathrm{Std}(\Lambda) \text{ of wreath type for } \Lambda \in \mathcal{L}_n(\alpha)\}.$$

The next Corollary should be compared with the results of Geetha and Goodman, [**15**], who show that $\mathcal{A}\wr\mathfrak{S}_m$ is a cellular algebra whenever $\mathcal{A}$ is a *cyclic* cellular algebra; by definition this means that all cell modules all cyclic.

COROLLARY 3.5. *We have that $\mathcal{BT}_n^{\alpha,wr}$ is a cellular basis for the subalgebra $\mathcal{H}_\alpha^{wr}(q)$ of $\mathcal{E}_n^\alpha(q)$, given by Lemma 20.*

PROOF. Let us first check that $\mathcal{BT}_n^{\alpha,wr}\subseteq\mathcal{H}_\alpha^{wr}(q)$. This is an argument similar to the one used in the beginning of Lemma 25. Supposing $\mathbb{s}=(\boldsymbol{\mathfrak{s}}\mid\mathbf{u})$ and $\mathbb{t}=(\mathbf{t}\mid\mathbf{v})$ are of wreath type we may use Lemma 23 to write

$$\begin{aligned} m_{\mathbb{s}\mathbb{t}} = &\quad g^*_{d(\boldsymbol{\mathfrak{s}})}\mathbb{E}_\Lambda\mathbb{B}^*_{d(\mathbf{u})}b_{\boldsymbol{\mu}}x_{\boldsymbol{\lambda}}\mathbb{B}_{d(\mathbf{v})}g_{d(\mathbf{t})} \\ = &\quad g_{y_{\boldsymbol{\mathfrak{s}}}}g^*_{d(\boldsymbol{\mathfrak{s}}_0)}\mathbb{E}_\Lambda\mathbb{B}^*_{d(\mathbf{u})}b_{\boldsymbol{\mu}}x_{\boldsymbol{\lambda}}\mathbb{B}_{d(\mathbf{v})}g_{d(\mathbf{t}_0)}g_{y_{\mathbf{t}}} \\ = &\quad \mathbb{B}_{y_{\boldsymbol{\mathfrak{s}}}}g^*_{d(\boldsymbol{\mathfrak{s}}_0)}\mathbb{E}_\Lambda\mathbb{B}^*_{d(\mathbf{u})}b_{\boldsymbol{\mu}}x_{\boldsymbol{\lambda}}\mathbb{B}_{d(\mathbf{v})}g_{d(\mathbf{t}_0)}\mathbb{B}_{y_{\mathbf{t}}} \end{aligned} \tag{3.56}$$

where $B_{y_{\boldsymbol{\mathfrak{s}}}},B_{y_{\mathbf{t}}}$ belong to $\mathfrak{S}_\Lambda^k$ and $\boldsymbol{\mathfrak{s}}_0,\mathbf{t}_0$ are multitableaux of the initial kind. Expanding $b_{\boldsymbol{\mu}}$ out as a linear combination of $\mathbb{B}_y$'s with $B_y\in\mathfrak{S}_\Lambda^m$ this becomes via Lemma 21 and Lemma 22 a linear combination of

$$\mathbb{B}_{y_{\boldsymbol{\mathfrak{s}}}}g^*_{d(\boldsymbol{\mathfrak{s}}_0)}\mathbb{E}_\Lambda\mathbb{B}^*_{d(\mathbf{u})}\mathbb{B}_y x_{\boldsymbol{\lambda}}\mathbb{B}_{d(\mathbf{v})}g_{d(\mathbf{t}_0)}\mathbb{B}_{y_{\mathbf{t}}}=\mathbb{B}_{y_{\boldsymbol{\mathfrak{s}}}}\mathbb{B}_{y_1}\mathbb{E}_\Lambda x_{\boldsymbol{\mathfrak{s}}_0\mathbf{t}_0}\mathbb{B}_{y_2}\mathbb{B}_{y_{\mathbf{t}}} \tag{3.57}$$

where $B_{y_1},B_{y_2}\in\mathfrak{S}_\Lambda^m$. Since $\boldsymbol{\mathfrak{s}}_0$ and $\mathbf{t}_0$ are of the initial kind we now get from Lemma 20 that (3.57), and hence also $m_{\mathbb{s}\mathbb{t}}$, belongs to $\mathcal{H}_\alpha^{wr}(q)$, as claimed.

Next it follows from Geetha and Goodman's results in [**15**], or via a direct counting argument, that the cardinality of $\mathcal{BT}_n^{\alpha,wr}$ is equal to the dimension of $\mathcal{H}_\alpha^{wr}(q)$. On the other hand, one easily checks that Lemma 24 holds for $\mathcal{BT}_n^{\alpha,wr}$ with respect to $h\in\mathcal{H}_\alpha^{wr}(q)$. Moreover, applying the straightening procedure of Lemma 25 on $m_{\mathbb{s}\mathbb{t}}$ for $\mathbb{s},\mathbb{t}$ tableaux of wreath type, the result is a linear combination of $m_{\mathbb{u}\mathbb{v}}$ where $\mathbb{u},\mathbb{v}$ are standard tableaux and still of wreath type. Thus the proof of Theorem 3.2 also gives a proof of the Corollary. □

REMARK 12. *Recall that we have $\alpha=(a_r^{k_r},\dots,a_1^{k_1})\in\mathcal{P}ar_n$ with $\mathfrak{S}_\Lambda^k=\mathfrak{S}_{k_1}\times\cdots\times\mathfrak{S}_{k_r}$. From Geetha and Goodman's cellular basis for $\mathcal{H}_\alpha^{wr}(q)$ one may have expected $\mathcal{BT}_n^{\alpha,wr}$ to be slightly different, namely given by pairs $(\boldsymbol{\mathfrak{s}}\mid\mathbf{u})$ such that $\boldsymbol{\mathfrak{s}}$ is a multitableau of the initial kind whereas $\mathbf{u}$ is an r-tuple of multitableaux on the numbers $\{a_ik_i\}$. For example for $\Lambda=\Big(\big((1,1),(2),(2),(2,1)\big)\mid\big((1),(1,1),(1)\big)\Big)$ we would have expected tableaux of the following form*

$$\mathbb{t}:=\left(\left(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}},\boxed{3\,|\,4},\boxed{5\,|\,6},\boxed{\begin{smallmatrix}7&9\\8&\end{smallmatrix}}\right)\middle|\left(\left(\boxed{2},\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}\right),\left(\boxed{4}\right)\right)\right) \tag{3.58}$$

where the shapes of the multitableaux occurring in $\mathbf{u}$ are given by the equally shaped tableaux of $\boldsymbol{\mathfrak{s}}$. On the other hand, there is an obvious bijection between our standard tableaux of wreath type and the standard tableaux appearing in Geetha and Goodman's basis and so the cardinality of our basis is correct, which is enough for the above argument to work.

4. $\mathcal{E}_n(q)$ is a direct sum of matrix algebras

In this section we use the cellular basis for $\mathcal{E}_n^{\alpha}(q)$ to show that $\mathcal{E}_n(q)$ is isomorphic to a direct sum of matrix algebras in the spirit of Lusztig and Jacon-Poulain d'Andecy's result for the Yokonuma-Hecke algebra.

Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$ and that $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ is a standard Λ-tableau. Recall the decomposition $d(\mathfrak{s}) = d(\mathfrak{s}_0)w_{\mathfrak{s}}$ such that $\mathfrak{s}_0$ is a multitableau of the initial kind and such that $\ell(d(\mathfrak{s})) = \ell(d(\mathfrak{s}_0)) + \ell(w_{\mathfrak{s}})$. We remark that if $\sigma \in \mathfrak{S}_\alpha$ permutes the numbers inside the components of $\mathfrak{s}$ then $w_{\mathfrak{s}} = w_{\mathfrak{s}\sigma}$. Indeed, for such σ we have that $d(\mathfrak{s}\sigma) = \sigma_0 d(\mathfrak{s})$ where $\sigma_0 \in \mathfrak{S}_n$ is an element permuting the numbers inside the components of $\mathbf{t}^{\lambda}$, that is $\mathbf{t}^{\lambda}\sigma_0$ is of the initial kind. But then $d(\mathfrak{s}\sigma) = (\sigma_0 d(\mathfrak{s}_0))w_{\mathfrak{s}}$ is the decomposition of $d(\mathfrak{s}\sigma)$ and so $w_{\mathfrak{s}} = w_{\mathfrak{s}\sigma}$, as claimed.

We now explain a small variation of this decomposition. Since $\mathfrak{s}$ is increasing we have that $i < j$ if and only if $\min(\mathfrak{s}^{(i)}) < \min(\mathfrak{s}^{(j)})$ whenever $\lambda^{(i)} = \lambda^{(j)}$. We now choose $B_y \in \mathfrak{S}_{\Lambda}^{k}$ such that $\bar{\mathfrak{s}} := \mathbf{t}^{\boldsymbol{\lambda}} B_y d(\mathfrak{s})$ is increasing in the *stronger* sense that $i < j$ iff $\min(\bar{\mathfrak{s}}^{(i)}) < \min(\bar{\mathfrak{s}}^{(j)})$ whenever $|\lambda^{(i)}| = |\lambda^{(j)}|$. Clearly such a B_y exists and is unique. We then consider the decomposition $d(\bar{\mathfrak{s}}) = d(\overline{\mathfrak{s}_0})w_{\bar{\mathfrak{s}}}$. Since $d(\bar{\mathfrak{s}}) = B_y d(\mathfrak{s})$ we have

$$d(\mathfrak{s}) = d(\mathfrak{s}_1)z_{\mathfrak{s}} \tag{4.1}$$

where $z_{\mathfrak{s}} := w_{\bar{\mathfrak{s}}}$ and where $\mathfrak{s}_1 := \mathbf{t}^{\boldsymbol{\lambda}} B_y^{-1} d(\bar{\mathfrak{s}}_0) = \bar{\mathfrak{s}}_0 B_y^{-1}$ is a tableau of wreath type. This gives us the promised decomposition of $d(\mathfrak{s})$. The numbers within the components of $\mathbf{t}^{\boldsymbol{\lambda}} z_{\mathfrak{s}}$ are just the numbers within the components of $\bar{\mathfrak{s}}$ and so $\mathbf{t}^{\boldsymbol{\lambda}} z_{\mathfrak{s}}$ is an increasing multitableau in the strong sense defined above.

LEMMA 26. *With the above notation we have the following properties.*

(1) *The decomposition in (4.1) is unique subject to $\mathfrak{s}_1$ being of wreath type and $\mathbf{t}^{\boldsymbol{\lambda}} z_{\mathfrak{s}}$ being increasing in the strong sense.*
(2) *The $z_{\mathfrak{s}}$'s appearing in (4.1) are representatives for distinct coset classes of $\mathfrak{S}_\Lambda \backslash \mathfrak{S}_n$ where $\mathfrak{S}_\Lambda$ is the stabilizer group of the set partition $A_{\boldsymbol{\lambda}}$, as introduced above.*

PROOF. Recall that $\mathfrak{S}_\Lambda$ is a product of groups $\left(\mathfrak{S}_{a_i} \times \cdots \times \mathfrak{S}_{a_i}\right) \rtimes \mathfrak{S}_{k_i}$. Let us prove (2). Supposing that $z_{\mathfrak{s}}$ and $z_{\mathfrak{t}}$ belong to the same $\mathfrak{S}_\Lambda$-coset, we have that $z_{\mathfrak{s}} = \sigma_0 B z_{\mathfrak{t}}$ where the components of σ_0 belong to the $\mathfrak{S}_{a_i} \times \cdots \times \mathfrak{S}_{a_i}$'s and the components of B belong to the $\mathfrak{S}_{k_i}$'s, according to the above description of $\mathfrak{S}_\Lambda$. Now both $\mathbf{t}^{\boldsymbol{\lambda}} z_{\mathfrak{s}}$ and $\mathbf{t}^{\boldsymbol{\lambda}} z_{\mathfrak{t}}$ are increasing multitableaux in the strong sense which implies that $B = 1$. Hence $\mathbf{t}^{\boldsymbol{\lambda}} z_{\mathfrak{s}}$ and $\mathbf{t}^{\boldsymbol{\lambda}} z_{\mathfrak{t}}$ are equal up to a permutation of the numbers inside their components and so, by the remark prior to the Lemma, we must have $z_{\mathfrak{s}} = z_{\mathfrak{t}}$. This proves part (2) of the Lemma. The uniqueness statement of (1) is shown by a similar argument. □

For any Λ-tableau $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ we define the Λ-tableau $\mathbb{s}_1$ via

$$\mathbb{s}_1 = (\mathfrak{s}_1 \mid \mathbf{u}). \tag{4.2}$$

LEMMA 27. *Suppose that* $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$, $\overline{\Lambda} = (\overline{\boldsymbol{\lambda}} \mid \overline{\boldsymbol{\mu}}) \in \mathcal{L}_n(\alpha)$, *that* $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ *is a standard* Λ*-tableau and that* $\mathbb{t} = (\mathbf{t} \mid \mathbf{v})$ *is a standard* $\overline{\Lambda}$*-tableau. Then we have that*

$$m_{\mathbb{t}^{\Lambda}\mathbb{s}} m_{\mathbb{t}\mathbb{t}^{\overline{\Lambda}}} = \begin{cases} m_{\mathbb{t}^{\Lambda}\mathbb{s}_1} m_{\mathbb{t}_1\mathbb{t}^{\overline{\Lambda}}} & \text{if } z_{\mathfrak{s}} = z_{\mathbf{t}} \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

PROOF. Both $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ are of type α and so $\mathbb{E}_{\Lambda} = \mathbb{E}_{\overline{\Lambda}}$. In the decomposition $d(\mathfrak{s}) = d(\mathfrak{s}_1) z_{\mathfrak{s}}$ from (4.1) we have in general that $l(d(\mathfrak{s})) \neq l(d(\mathfrak{s}_1)) + l(z_{\mathfrak{s}})$, but even so $m_{\mathbb{t}^{\Lambda}\mathbb{s}} = m_{\mathbb{t}^{\Lambda}\mathbb{s}_1} g_{z_{\mathfrak{s}}}$ by Lemma 23. Similarly we have that $m_{\mathbb{t}^{\overline{\Lambda}}\mathbb{t}} = m_{\mathbb{t}^{\overline{\Lambda}}\mathbb{t}_1} g_{z_{\mathbf{t}}}$. Hence we get via Proposition 3.1 that

$$\begin{aligned} m_{\mathbb{t}^{\Lambda}\mathbb{s}} m_{\mathbb{t}\mathbb{t}^{\overline{\Lambda}}} &= m_{\mathbb{t}^{\Lambda}\mathbb{s}_1} g_{z_{\mathfrak{s}}} g^*_{z_{\mathbf{t}}} m_{\mathbb{t}_1\mathbb{t}^{\Lambda}} = m_{\mathbb{t}^{\Lambda}\mathbb{s}_1} \mathbb{E}_{\Lambda} g_{z_{\mathfrak{s}}} g^*_{z_{\mathbf{t}}} \mathbb{E}_{\Lambda} m_{\mathbb{t}_1\mathbb{t}^{\Lambda}} \\ &= m_{\mathbb{t}^{\Lambda}\mathbb{s}_1} g_{z_{\mathfrak{s}}} \mathbb{E}_{(A_{\boldsymbol{\lambda}}) z_{\mathfrak{s}}} \mathbb{E}_{(A_{\boldsymbol{\lambda}}) z_{\mathbf{t}}} g^*_{z_{\mathbf{t}}} m_{\mathbb{t}_1\mathbb{t}^{\Lambda}}. \end{aligned} \tag{4.4}$$

We now apply the previous Lemma to deduce that $\mathbb{E}_{(A_{\boldsymbol{\lambda}}) z_{\mathfrak{s}}} \mathbb{E}_{(A_{\boldsymbol{\lambda}}) z_{\mathbf{t}}} = 0$ if $z_{\mathfrak{s}} \neq z_{\mathbf{t}}$ and hence also $m_{\mathbb{t}^{\Lambda}\mathbb{s}} m_{\mathbb{t}\mathbb{t}^{\overline{\Lambda}}} = 0$ if $z_{\mathfrak{s}} \neq z_{\mathbf{t}}$, thus showing the second part of the Lemma. Finally, if $z_{\mathfrak{s}} = z_{\mathbf{t}}$ we have that

$$g_{z_{\mathfrak{s}}} \mathbb{E}_{(A_{\boldsymbol{\lambda}}) z_{\mathfrak{s}}} \mathbb{E}_{(A_{\boldsymbol{\lambda}}) z_{\mathbf{t}}} g^*_{z_{\mathbf{t}}} = \mathbb{E}_{\Lambda} g_{z_{\mathfrak{s}}} g^*_{z_{\mathbf{t}}} \mathbb{E}_{\Lambda} = \mathbb{E}_{\Lambda} \tag{4.5}$$

as can be seen, once again, by expanding $z_{\mathfrak{s}}$ out in terms of simple transpositions and noting that the action at each step involves different blocks. The first part of the Lemma now follows by combining (4.4) and (4.5). □

Recall that for any algebra $\mathcal{A}$ we denote by $\mathrm{Mat}_N(\mathcal{A})$ the algebra of $N \times N$-matrices with entries in $\mathcal{A}$.

The cardinality of $\{z_{\mathfrak{s}}\}$ is $b_n(\alpha)$, the Faà di Bruno coefficient. We introduce an arbitrary total order on $\{z_{\mathfrak{s}}\}$ and denote by $M_{\mathfrak{s}\mathbf{t}}$ the elementary matrix of $\mathrm{Mat}_{b_n(\alpha)}\big(\mathcal{H}^{wr}_{\alpha}(q)\big)$ which is equal to 1 at the intersection of the row and column indexed by $z_{\mathfrak{s}}$ and $z_{\mathbf{t}}$, and 0 otherwise.

We can now prove our promised isomorphism Theorem.

THEOREM 3.3. *Let* α *be a partition of* n*. The* S*-linear map* Ψ_{α} *given by*

$$\mathcal{E}^{\alpha}_n(q) \longrightarrow \mathrm{Mat}_{b_n(\alpha)}\big(\mathcal{H}^{wr}_{\alpha}(q)\big), \; m_{\mathbb{s}\mathbb{t}} \mapsto m_{\mathbb{s}_1\mathbb{t}_1} M_{\mathfrak{s}\mathbf{t}}$$

is an isomorphism of S*-algebras. A similar statement holds for the specialized algebra over* $\mathbb{K}$.

PROOF. Note first that by Corollary 3.5 we have that $m_{\mathbb{s}_1\mathbb{t}_1} \in \mathcal{H}^{wr}_{\alpha}(q)$. Furthermore, by the uniqueness statement of the previous Lemma we have that Ψ_{α} maps an S-basis to an S-basis and so we only need to show that it is a homomorphism, preserving the multiplications on both sides.

For this suppose that $\Lambda, \overline{\Lambda} \in \mathcal{L}_n(\alpha)$. Given a pair of standard Λ-tableaux $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u}_1)$, $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}_2)$ and a pair of standard $\overline{\Lambda}$-tableaux $\mathbb{u} = (\mathfrak{u} \mid \mathbf{v}_1)$, $\mathbb{v} = (\mathfrak{v} \mid \mathbf{v}_2)$ we have by the previous Lemma that

$$\begin{aligned} m_{\mathbb{s}\mathbb{t}} m_{\mathbb{u}\mathbb{v}} &= \begin{cases} m_{\mathbb{s}\mathbb{t}_1} m_{\mathbb{u}_1\mathbb{v}} & \text{if } z_{\mathbf{t}} = z_{\mathfrak{u}} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} g^*_{z_{\mathfrak{s}}} (m_{\mathbb{s}_1\mathbb{t}_1} m_{\mathbb{u}_1\mathbb{v}_1}) g_{z_{\mathfrak{v}}} & \text{if } z_{\mathbf{t}} = z_{\mathfrak{u}} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.6}$$

Expanding $m_{\mathbb{s}_1\mathbb{t}_1} m_{\mathbb{u}_1\mathbb{v}_1} \in \mathcal{H}^{wr}_\alpha(q)$ out as a linear combination of cellular basis elements $m_{\mathbb{a}_1\mathbb{b}_1}$ of $\mathcal{H}^{wr}_\alpha(q)$ we have that $m_{\mathbb{s}\mathbb{t}} m_{\mathbb{u}\mathbb{v}}$ is the corresponding linear combination of $g^*_{z_\mathfrak{s}} m_{\mathbb{a}_1\mathbb{b}_1} g_{z_\mathfrak{v}}$'s, and so

$$\Psi_\alpha(m_{\mathbb{s}\mathbb{t}} m_{\mathbb{u}\mathbb{v}}) = \begin{cases} m_{\mathbb{s}_1\mathbb{t}_1} m_{\mathbb{u}_1\mathbb{v}_1} M_{\mathfrak{s}\mathfrak{v}} & \text{if } z_\mathfrak{t} = z_\mathfrak{u} \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

On the other hand, by the matrix product formula $M_{\mathfrak{s}\mathfrak{t}} M_{\mathfrak{u}\mathfrak{v}} = \delta_{z_\mathfrak{t} z_\mathfrak{u}} M_{\mathfrak{s}\mathfrak{v}}$ we have that

$$\Psi_\alpha(m_{\mathbb{s}\mathbb{t}})\Psi_\alpha(m_{\mathbb{u}\mathbb{v}}) = \begin{cases} m_{\mathbb{s}_1\mathbb{t}_1} m_{\mathbb{u}_1\mathbb{v}_1} M_{\mathfrak{s}\mathfrak{v}} & \text{if } z_\mathfrak{t} = z_\mathfrak{u} \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Comparing (4.7) and (4.8) we conclude that Ψ_α is an algebra homomorphism as claimed. The Theorem is proved. □

EXAMPLE 6. *For $n = 4$ we have that*

Partition of 4	*Faà di Bruno coeff.*	$\mathcal{H}^{wr}$	$\dim \mathcal{E}_4^\alpha(q)$
(1^4)	1	$\mathcal{H}_1(q) \wr \mathfrak{S}_4 \ (\cong S\mathfrak{S}_4)$	24
$(2,1^2)$	6	$\mathcal{H}_2(q) \wr \mathfrak{S}_1 \otimes \mathcal{H}_1(q) \wr \mathfrak{S}_2 \ (\cong \mathcal{H}_2(q) \otimes S\mathfrak{S}_2)$	144
(2^2)	3	$\mathcal{H}_2(q) \wr \mathfrak{S}_2$	72
$(3,1)$	4	$\mathcal{H}_3(q) \wr \mathfrak{S}_1 \otimes \mathcal{H}_1(q) \wr \mathfrak{S}_1 \ (\cong \mathcal{H}_3(q))$	96
(4)	1	$\mathcal{H}_4 \wr \mathfrak{S}_1 \ (\cong \mathcal{H}_4(q))$	24

Thus,

$$\mathcal{E}_4(q) \cong S\mathfrak{S}_4 \oplus \mathrm{Mat}_6\left(\mathcal{H}_2(q) \otimes S\mathfrak{S}_2\right) \oplus \mathrm{Mat}_3\left(\mathcal{H}_2(q) \wr \mathfrak{S}_2\right) \oplus \mathrm{Mat}_4\left(\mathcal{H}_3(q)\right) \oplus \mathcal{H}_4(q).$$

Note that summing up the dimensions of the last column we get $360 = 4!b_n(4) = 24 \times 15$, *which is the dimension of* $\mathcal{E}_4(q)$ *as expected.*

Throughout the thesis we adopt the following conventions:

- We use the normal frak font, like $\mathfrak{s}$, to denote tableaux whose shape is a composition.
- We use the boldfrak font, like $\boldsymbol{\mathfrak{s}}$, to denote multitableaux whose shape is a multicomposition.
- We use the mathematical doble-struck font, like $\mathbb{s}$, to denote tableaux whose shape is an element of $\mathcal{L}_n$.
- For λ a composition (resp. $\boldsymbol{\lambda}$ a multicomposition, resp. Λ an element of $\mathcal{L}_n$) we denote by $\mathfrak{t}^{\lambda}$ (resp. $\boldsymbol{\mathfrak{t}}^{\boldsymbol{\lambda}}$, resp. $\mathbb{t}^{\Lambda}$) the maximal tableau of shape λ (resp. shape $\boldsymbol{\lambda}$, resp. shape Λ) as introduced in the text. Note that $\boldsymbol{\mathfrak{t}}^{\boldsymbol{\lambda}}$ and $\mathbb{t}^{\Lambda}$ are not the unique maximal tableaux of their shape.
- For λ a composition (resp. $\boldsymbol{\lambda}$ a multicomposition, resp. Λ an element of $\mathcal{L}_n$) we denote by $\mathrm{Std}(\lambda)$ (resp. $\mathrm{Std}(\boldsymbol{\lambda})$, resp. $\mathrm{Std}(\Lambda)$) the set of standard λ-tableaux (resp. $\boldsymbol{\lambda}$-tableaux, resp. Λ-tableaux).

List of Notations

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