Speed selection and stability of wavefronts for delayed monostable reacction-diffusion equations

by

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CHAPTER I

Introduction

In this work, we study the asymptotic convergence of solution $u(t,x)$ of the initial value problem for a monostable reaction-diffusion equation with delayed reaction

$$(1.1) \qquad u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)),$$

$$(1.2) \qquad u(s,x) = w_0(s,x), \ (s,x) \in \Pi_0 := [-h,0] \times \mathbb{R}.$$

In the sequel, it is always assumed that the continuous function $w_0(s,x)$ is locally Hölder continuous in $x \in \mathbb{R}$, uniformly with respect to s, and that the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the monostability condition

(H) the equation $g(x) = x$ has exactly two nonnegative solutions: 0 and $\kappa > 0$. Moreover, g is C^1-smooth in some δ_0-neighborhood of the equilibria where $g'(0) > 1$, $g'(\kappa) < 1$, and also satisfies the Lipschitz condition $|g(u) - g(v)| \le L_g|u - v|, \ u, v \in [0, \kappa]$. In addition, there are $C > 0, \ \theta \in (0,1]$, such that $|g'(u) - g'(0)| + |g'(\kappa) - g'(\kappa - u)| \le Cu^\theta$ for $u \in (0, \delta_0]$. Without restricting generality, we will also assume that g is linearly and C^1-smoothly extended on $(-\infty, 0]$.

Equation (1.1) (together with its non-local versions) is an important model in the population dynamics [8, 9, 19, 21, 25, 27, 28, 37, 39, 58, 61, 62] where it is used to describe the spatio-temporal evolution of a single-species population. In this interpretation of (1.1), g is a birthrate function, $u(t,x)$ denotes the population density at location x and time t, and it is supposed that the species reaches sexual maturity at age $h > 0$. Clearly, the Cauchy problem (1.1), (1.2) can be solved by the method of steps [17], where in the first step we have to look for the solution of the inhomogeneous linear equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(w_0(t-h,x)),\ t \in [0,h],\ x \in \mathbb{R},$$

satisfying the initial condition $u(0,x) = w_0(0,x)$. Besides the hypothesis **(H)**, from a biological point of view, it is realistic to assume that the birth function g is either strictly increasing or unimodal (i.e. g has exactly one critical point which is the absolute maximum point [25, 54, 61]) function on $\mathbb{R}_+$. In the population dynamics, equation (1.1) improves certain weaknesses (cf. [22] or [56, pp. 56-58]) of the logistic growth model given by the KPP-Fisher delayed or nonlocal equations [4, 6, 10, 13, 20, 24]. One of the most interesting features of the dynamics in (1.1) is the existence of smooth positive solutions $u(t,x) = \phi(x+ct)$ satisfying the boundary conditions $\phi(-\infty) = 0$ and $\liminf_{t\to+\infty} \phi(t) > 0$ (for $c > 0$, cf. [19]). Such solutions are called traveling semi-wavefronts (or wavefronts if additionally $\phi(+\infty) = \kappa$), they describe waves of colonisation propagating with the velocity c. The convergence and stability properties of wavefronts to (1.1) are quite well understood in the non-delayed case (i.e. when $h = 0$). The studies of the front stability in non-delayed monostable equation (1.1) were initiated in 1976 by Sattinger [48] (see [41] for the state-of-art on this topic), but already the seminal work of Kolmogorov, Petrovskii, Piskunov (1937) presented a first deep analysis of the convergence of the solution $u(t,x)$ of (1.1), (1.2) (with $-u + g(u) = u(1-u)$ and with $w_0(s,x)$ being the Heaviside step function $H(x)$) to a monotone wavefront.

Now, the investigation of asymptotic behavior of solution to problem (1.1), (1.2) becomes a much more challenging task when $h > 0$. For instance, the recent works [10, 11, 19, 20, 24, 28, 54] show that the delay h has a strong influence on the geometry of front's profile ϕ and complicates enormously the studies of the front uniqueness [1, 8, 55, 58] and stability [8, 25, 27, 28, 29, 37, 39, 41, 58]. Moreover, in order to be able to perform the local stability analysis of equation (1.1), it was always necessary to assume the additional sub-tangency restriction

$$g(u) \leq g'(0)u, \quad u \geq 0. \tag{1.3}$$

Under this assumption, all wavefronts of equation (1.1) are known as *'pulled'* fronts (see [6, 18, 44, 45, 46, 52, 60] for further details), model (1.1) is linearly determined [23, 59] and there exists a positive number $c_* > 0$ (called the minimal speed of propagation) separating the positive axis on the set of admissible *semi-wavefronts* speeds $[c_*, +\infty)$ and the set $[0, c_*)$ of velocities c for which does not exist *any* non-constant positive bounded wave solution $u(t,x) = \phi(x + ct)$ [19]. Furthermore, the minimal speed c_* is determined from the characteristic equation

$$\chi(z, c) := z^2 - cz - 1 + g'(0)e^{-zch} = 0 \tag{1.4}$$

as the unique real value $c_\#$ for which $\chi(z, c)$ has a positive double zero $\lambda_1(c_\#) = \lambda_2(c_\#)$ (i.e. c_* is equal to $c_\#$ if (1.3) holds). Note that for $c > c_\#$ equation $\chi(z, c) = 0$ has exactly two positive simple roots, we will denote them as $\lambda_1(c) < \lambda_2(c)$.

In this way, as far as we know, all studies of wave's stability in the delayed model (1.1) have dealt exclusively with the stability of pulled wavefronts. Nevertheless, from an ecological point of view, models with the birth functions which are not sub-tangential at $u = 0$ are also quite interesting in view of the interpretation of non-sub-tangentiality property of g in terms of a weak Allee effect [6, 16, 44]. In the non-delayed case, it is well known [18, 44, 45, 46, 52, 60] that such systems can possess a special type of minimal wavefronts called the *'pushed'* fronts. As the characterising property of a pushed wave for model (1.1), we can take the following one: the minimal wavefront $u(t,x) = \phi(x + c_* t)$ is pushed if the velocity c_* is not linearly determined, i.e. if $c_* > c_\#$. The recent work [44] explains why, contrarily to the pulled waves, the pushed colonisation waves can be considered as waves promoting genetic diversity in the ecological systems.

Respect to the selection problem,the concept of 'speed selection' reflects the evident fact that the properties of w_0 may determine the speed of propagation of the initial 'concentration' (of something) $w_0(s,x)$ from the right side of the x-axis $\mathbb{R}$ (where w_0 is separated from 0) to the left side of $\mathbb{R}$ (where w_0 vanishes). Moreover, in the non-delayed case (when $h = 0$) it is well known [46] that, given a converging solution $u(t,x,w_0) \rightsquigarrow \phi_{w_0}(x+ct)$, the speed of propagation c 'choosen' by $u(t,x,w_0)$ depends mainly only on the asymptotic behavior of $w_0(s,x)$ at $x = -\infty$. It is clear also that the speed selection problem is closely related to the front stability question: indeed, if some wavefront $u = \phi(x + c_0 t)$ is stable (in an appropriate metric phase space), then each initial datum $w_0(s,x)$ close to $\phi(x + c_0 s)$ yields a 'concentration' distribution $u(t,x)$ propagating to the left of $\mathbb{R}$ with the same velocity c_0. Below we will give precise mathematical formulations for the above informal discussion. So, we can to show that there exist a critical value $\lambda_* > 0$ so that if the behavior in $x \to -\infty$ of the initial date $w_0(s,x)$ is as $Ae^{-\lambda x}$(for some $A > 0$ and uniformly in $s \in [-h,0]$) with $\lambda < \lambda_*$, then the solution generated by w_0, $u(t,x)$, converges uniformly on $\mathbb{R}$ to some wavefront $\phi(x+ct)$, with some $c > c_*$ to specific, which has the same behavior than w_0 in $x \to -\infty$. While that for $\lambda > \lambda_*$ if the behavior for $x \to -\infty$ of the initial date is as $A(x)e^{\lambda x}$(for some non negative and bounded function $A(x)$ and uniformly in $s \in [-h,0]$) with $\lambda > \lambda_*$ then the solution $u(t,x)$ converges uniformly on $\mathbb{R}$ to a minimal wavefronts $\phi(x + c_* t)$ which have not the same behavior than w_0 necessarily. To precise this ideas we divide the text in two part to analyze these two cases in more detail and in there we dedicated more comments respect to the problems of stability and selection.

In the first part, we study the stability of minimal wave front in pushed case. The study of pushed waves in the monostable delayed model (1.1) was initiated in [27, 55] (curiously, in the first work [49] dealing with traveling waves in delayed models, all waves were tacitly presumed to be pulled). In [55], after assuming monotonicity of g, it was proved that the unique minimal wavefront propagating with the speed $c_* > c_\#$ must have a strictly increasing profile ϕ with the following asymptotic representation at $-\infty$:

$$\phi(t + s_0) = e^{\lambda_2 t} + O(e^{(\lambda_2+\varsigma)t}),\ \lambda_2 := \lambda_2(c_*),\ \varsigma > 0, \quad t \to -\infty. \tag{1.5}$$

It should be noted that the situation when non-monotone (for example, unimodal) birth function $g : [0, \kappa] \to \mathbb{R}_+$ does not satisfy (1.3) is not completely understood till now. In fact, even the existence of the minimal speed of propagation c_*, as the lowest value from a closed connected unbounded set of all admissible wavefront (or semi-wavefront [4, 19]) velocities, is not yet proved for the case of non-monotone and not sub-tangential g. From the formal point of view, the existence of the pushed fronts to the delayed model (1.1) neither was established in [55]. In any case, this point can be easily completed:

Proposition 1. *Assume that $u = \phi(x + c_* t)$, $c_* > c_\#(h_0)$, is a pushed traveling front to the monotone model (1.1) considered with some fixed $h_0 \geq 0$. Then there exists a positive δ such that equation (1.1) possesses a pushed traveling front for each non-negative $h \in (h_0 - \delta, h_0 + \delta)$. In particular, there exists a delayed equation (1.1)* *with $h > 0$ possessing the minimal monotone wavefront $u = \phi(x+c_*t)$ with the profile ϕ satisfying the asymptotic formula (1.5).*

Proof. Since $c_{\#}(h)$ depends continuously on $h \geq 0$, the first part of Proposition 1 will be proved if we establish the lower semicontinuity of $c_*(h)$ at h_0. Then the existence of pushed wavefronts to the equation (1.1) considered with small positive delays follows from the existence of the pushed wavefronts to the Fisher type population genetic model [23, Theorem 11] $u_t(t,x) = u_{xx}(t,x) - u(t,x) + (10u(t,x) + 3u^2(t,x) - 5u^3(t,x))/8$. Hence, it suffices to prove the following

Claim. *Suppose that $h_j \to h_0,\ c_*(h_j) \to c_0$ as $j \to +\infty$. Then $c_0 \geq c_*(h_0)$.*

Indeed, take some $c > c_0$. Then, for all sufficiently large j, the equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h_j,x))$$

has a unique (up to translation) positive strictly monotone wavefront $u(t,x) = \phi_j(x+ct)$. Without the loss of the generality, we can assume that $\phi_j(0) = \kappa/2$. It is easy to see (cf. [55]) that each profile ϕ_j satisfies the integral equation

$$(1.6)\quad \phi(t) = \frac{1}{\xi_2 - \xi_1}\left(\int_{-\infty}^{t} e^{\xi_1(t-s)} g(\phi(s-ch_j))ds + \int_t^{+\infty} e^{\xi_2(t-s)} g(\phi(s-ch_j))ds\right),$$

where $\xi_1 < 0 < \xi_2$ are roots of the equation $z^2 - cz - 1 = 0$. Since $|\phi_j'(t)| \leq \kappa/\sqrt{c^2+4}$, $|\phi_j(t)| \leq \kappa$, the sequence ϕ_j has a subsequence ϕ_{j_k} wich converges, uniformly on compact subsets of $\mathbb{R}$, to the monotone continuous bounded function $\phi_0(t),\ \phi_0(0) = \kappa/2$. By the Lebesgues dominated convergence theorem, ϕ_0 satisfies the equation (1.6) with h_0 and therefore ϕ_0 is a positive profile of strictly monotone wavefront propagating with the velocity c [19, 55]. In this way, $c \geq c_*(h_0)$ for every $c > c_0$ that yields $c_0 \geq c_*(h_0)$. $\square$

Formula (1.5) implies that pushed profiles $\phi(s)$ converges to 0 at $-\infty$ more rapidly than the profiles of other (i.e. non-minimal or pulled) waves behaving as

$$\phi(t+s_0) = (-t)^m e^{\lambda_1 t} + O(e^{(\lambda_1+\varsigma)t}),\ \lambda_1 := \lambda_1(c),\ \varsigma > 0,\ m \in \{0,1\}, \quad t \to -\infty.$$

The fast asymptotic decay of pushed fronts at $-\infty$ makes them similar to the so-called *bistable* fronts [14, 50, 60]. Actually, by analysing the inside dynamics of wavefronts, Garnier *et al* [18] (in the non-delayed case) and Bonnefon *et al* [6] (in the delayed case) have recently proposed a general definition of pushed waves which allows to consider the monostable pushed fronts and the bistable fronts within a unified framework. An additional argument if favor of this insight is provided by the theory of nonlinear stability of waves. Indeed, both monostable pushed fronts and bistable fronts are proved to have rather good stability properties [14, 42, 45, 50, 52, 53]. Furthermore, the most complete and comprehensible proof of the asymptotic stability of monostable pushed front given in [45] uses constructions and results obtained for a bistable model in [14].

Hence, the main aim of this part is to study the stability properties of monostable pushed fronts to the monotone delayed model (1.1). We are going to achieve this goal by developing several ideas and methods from [14, 42, 45, 55]. We also will establish the asymptotic convergence of solutions for the initial value problem (1.1), (1.2) to an appropriate pushed wavefront when, in addition to **(H)**, g is monotone and when w_0 satisfies, for some $A, B > 0$, $\sigma \in (0, \kappa)$ and $\mu > \lambda_1(c_*)$ the following conditions **(IC)**:

$(IC1)$ $\quad 0 \le w_0(s,x) \le \kappa, \quad (s,x) \in \Pi_0$

$(IC2)$ $\quad w_0(s,x) \le Ae^{\mu x}, \quad (s,x) \in \Pi_0;$

$(IC3)$ $\quad w_0(s,x) > \kappa - \sigma, \quad s \in [-h, 0], \quad x \ge B.$

Alternately, we consider the similar but weaker conditions **(IC')** :

$(IC1')$ $\quad 0 \leq w_0(s,x) \leq |w_0|_\infty := \sup_{(s,x)\in\Pi_0} < \infty, \quad (s,x) \in \Pi_0$

$(IC2')$ $\quad \liminf_{x\to\infty} \min_{s\in[-h,0]} w_0(s,x) > 0.$

From the monotonicity of g and the hypotheses **(H)**, **(IC)**, by invoking the well-known existence and uniqueness results and the comparison principle [15, Chapter 1, Theorems 12, 16], we can deduce the existence of a unique classical solution $u = u(t,x) : [-h,+\infty) \times \mathbb{R} \to [0,\kappa]$ to (1.1), (1.2) (i.e. of a continuous bounded function u having continuous derivatives u_t, u_x, u_{xx} in $\Omega = (0,+\infty)\times\mathbb{R}$ and satisfying (1.1) in Ω as well as (1.2) in Π_0). As the following proposition shows, the asymptotic behavior of this solution $u(t,x)$ on bounded subsets of $\mathbb{R}$ is quite simple:

Proposition 2. *Suppose that the initial datum* $w_0 \not\equiv 0$ *satisfies* $(IC1)$ *and that the Lipshitz continuous map* $g : [0,\kappa] \to [0,\kappa]$ *has exactly two fixed points:* 0 *and* $\kappa > 0$. *Then* $\lim_{t\to\infty} u(t,x) = \kappa$ *uniformly on compact subsets of* $\mathbb{R}$.

At first glance, if additionally we assume the monotonicity of g, Proposition 2 seems to follow from quite general results on spreading speeds to continuous-time semiflows established in [26, 27]. Indeed, [27, Theorem 34] shows that even rather weak positivity condition assumed in Proposition 2 is enough to assure stronger convergence

$$\lim_{t\to\infty} \sup_{x\in[-c't,c't]} |u(t,x) - \kappa| = 0, \; c' \in (0,c_*), \tag{1.7}$$

once g is a subhomogeneous function: $\rho g(x) \leq g(\rho x)$ for all $\rho \in [0,1]$ and $x \geq 0$. It is easy to see, however, that the latter condition implies the sub-tangency inequality (1.3).

Our proof of Proposition 2 follows closely the main lines of [2], where Aronson and Weinberger established a similar result for non-delayed equations. See also [62, Theorem 3.2] for an analogous assertion proved for a non-diffusive delay differential equation with spatial non-locality in an unbounded domain. In general (e.g. under condition $(IC2)$) the convergence of $u(t,\cdot) \to \kappa$, $t \to +\infty$, is not uniform on $\mathbb{R}$: this is an immediate outcome of our subsequent investigation of the asymptotic behavior of the *entire solution* $u(t,x)$ as $t \to +\infty$ on *the whole real* x*-line* $\mathbb{R}$.

In order to state the main results of this part, we take a pushed front $\phi(x+c_*t)$ for equation (1.1) and fix a positive number $\lambda < \mu$ such that $\lambda \in (\lambda_1(c_*), \lambda_2(c_*))$. We will also consider the Banach space

$$C_\lambda(\mathbb{R}) = \left\{ y \in C(\mathbb{R},\mathbb{R}) : |y|_\lambda := \max\{\sup_{x\le 0} e^{-\lambda x}|y(x)|, \sup_{x\ge 0}|y(x)|\} < \infty \right\}.$$

This norm will use along the text still in the case when $\lambda \in (\lambda_1(c), \lambda_2(c))$ where $c > c_*$.

Observe that $|y|_\lambda = \sup_{x\in\mathbb{R}} |y(x)|/\eta(x)$, where $\eta(x) := \min\{e^{\lambda x}, 1\}$. Our first theorem shows that the pushed front $\phi(x+c_*t)$, $c_* > c_\#$, is nonlinearly stable with asymptotic phase [47]:

Theorem I.1. *Let g be monotone and conditions* (**IC**), (**H**) *be satisfied. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\phi(\cdot+c_*s) - w_0(s,\cdot)|_\lambda < \delta$, $s \in [-h,0]$, implies that $|\phi(\cdot+c_*t) - u(t,\cdot)|_\lambda < \epsilon$ for all $t \ge 0$. Here $u(t,x)$ is solution of the initial value problem (1.1), (1.2). Furthermore, there exists s_0 such that $|\phi(\cdot+c_*t+s_0) - u(t,\cdot)|_\lambda \to 0$ as $t \to +\infty$.*

The stability result of Theorem I.1 follows from Corollary 13 proved in Section 2 while the asymptotic convergence $u(t,x) \to \phi(x+c_*t+s_0)$, $t \to +\infty$, follows from the next theorem. It describes the global stability properties of the pushed fronts with respect to initial data satisfying the hypothesis (**IC**):

Theorem I.2. *Let g be monotone and conditions* (**IC**), (**H**) *be satisfied. Then the solution $u(t,x)$ of the initial value problem (1.1), (1.2) asymptotically converges to a shifted front. In fact, for some $s_0 \in \mathbb{R}$,*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(t,x) - \phi(x + c_*t + s_0)|/\eta(x + c_*t) = 0. \tag{1.8}$$

It is instructive to compare Theorems I.1, I.2 with stability results obtained for non-critical pulled fronts in the delayed model (1.1) with monotone reaction g satisfying (1.3) and **(H)**. For example, taking initial functions w_0 satisfying ($IC1$) and assuming that the initial disturbance $\phi(\cdot + cs) - w_0(s,\cdot)$ belongs to the weighted Sobolev space $H^1_{\eta^2}(\mathbb{R})$ and depends continuously on $s \in [-h, 0]$, Mei et al [37, Theorem 2.2] proved that $|\phi(\cdot + ct) - u(t,\cdot)|_0 \to 0$ exponentially when $t \to +\infty$. Hence, in view of the continuous imbedding $H^1_{\eta^2}(\mathbb{R}) \subset C_\lambda(\mathbb{R}) \cap C^{0,1/2}(\mathbb{R}_+)$, initial functions $w_0(s,x)$ in [37] are uniformly Hölder continuous in x and converge at $+\infty$, $w_0(s,+\infty) = \kappa$ (in fact, this convergence is uniform in $s \in [-h,0]$, so that each w_0 meets trivially the restriction (IC3)). They should also satisfy, for some $C > 0$ the inequality

$$|\phi(x + cs) - w_0(s,x)| \le Ce^{\lambda x}, \ (s,x) \in \Pi_0, \ \lambda \in (\lambda_1, \lambda_2). \tag{1.9}$$

Due to the asymptotic representation (1.5) and to certain freedom in the choice of λ, μ, in the case of pushed fronts, the latter condition amounts precisely to the hypothesis (IC2). Nevertheless, in contrast to inequality (1.9) considered with a pushed front $u = \phi(x + c_*s)$, the same inequality considered with a pulled front $u = \phi(x + cs)$ is not satisfied if we take the Heaviside step function $H(x)$ as the initial function $w_0(s,x) = H(x)$. Thus the question about the asymptotic form of solution $u(t,x)$ to the Cauchy problem (1.1), (1.2) with $w_0(s,x) = H(x)$ and with the sub-tangential g still remains unanswered in the delayed case. It is worth to recall that precisely this question formulated for a non-delayed monostable equation (1.1) was the main object of studies in the seminal work by Kolmogorov, Petrovskii, Piskunov in 1937.

Now, it is worth noticing that equation (1.1) is invariant wit respect to the transformation $x \to -x$ so that the statements of Theorems I.1 and I.2 can be easily adapted to the case when the initial function $w_0(s,-x)$ meets the hypothesis (**IC**). Evidently, in such a case, we should use *pushed backs* of the form $u = \phi(-x + c_*t)$ instead of the pushed wavefronts. Then the natural question is whether solution $u(t,x)$ converges to a combination of a pushed front and a pushed back when the both non-zero functions $w_0(s,x), w_0(s,-x)$ satisfy conditions (IC1), (IC2). In particular, this happens when w_0 has compact support. To the best of our knowledge, the studies of the asymptotic form of solutions to the monostable reaction-diffusion equations having compactly supported initial data were initiated in [2, 45, 53, 57]. Here, we analyse a similar problem in the presence of delay; hence, our third theorem considers the initial data for (1.1), (1.2) exponentially vanishing at both infinities.

Theorem I.3. *Assume that* $u = \phi(x + c_*t)$, $c_* > c_{\#}$, *is a pushed traveling front to equation (1.1). If non-zero functions* $w_0(s,x), w_0(s,-x)$ *satisfy conditions (IC1), (IC2) then the solution* $u = u(t,x)$ *of the initial value problem (1.1), (1.2) asymptotically converges to a combination of two shifted fronts, i.e. for some* $s_1, s_2 \in \mathbb{R}$,

$$\lim_{t\to\infty} \sup_{x \le 0} |u(t,x) - \phi(x + c_*t + s_1)|/\eta(x + c_*t) = 0,$$
$$\lim_{t\to\infty} \sup_{x \ge 0} |u(t,x) - \phi(-x + c_*t + s_2)|/\eta(-x + c_*t) = 0.$$

Clearly, Theorem I.3 combined with the comparison principle shows that relation (1.11) holds for each solution $u = u(t,x)$ to (1.1) once associated initial datum

$w_0(s,x) \not\equiv 0$ satisfies (IC1). Moreover, since Theorem I.3 implies that

$$\lim_{t\to\infty} \sup_{x\notin(-c't,c't)} u(t,x) = 0, \ c' > c_*,$$

we can conclude that the speed c_* of pushed waves coincides with the spreading speed for model (1.1). Without restriction (1.3), this important result was for the first time established in [26, 27] (in a much more general setting). Therefore Theorem I.3 can be also viewed as an essential improvement of the mentioned Liang and Zhao result for the particular case of Eq. (1.1).

As in [14, 45], the method of sub- and super-solutions is a key tool for proving our main results. The sub- and super-solutions will be obtained as suitable deformations (invented by Fife and McLeod in [14] for the bistable systems and adapted by Rothe in [45] for the monostable equations) of the pushed wavefront. The other important idea exploited in [14, 45] is the use of an appropriate Lyapunov functional for proving the wave stability. However, the construction of such a functional seems to be a rather difficult task in the case of the functional differential equation (1.1). Thus, instead of this, we decided to use the Berestycki and Nirenberg method of the sliding solutions [5, 55] as well as some ideas of the approach developed by Ogiwara and Matano in [42]. It is natural to expect that the rate of convergence in (1.8) is exponential, see e.g. [14, 37, 39, 45, 47]. The demonstration of this fact, however, is based on a different approach and will be considered in a separate work.

In the second part we continue our study about stability, but we consider the super critical fronts ($c > c_\#$). Also, in this part, we can to solve the selection problem. The main difference with the previous works consists in generally non-convex and non-smooth nature of the monotone birth function g: in fact, we do not even require the subtangency condition

$$g(x) \leq g'(0)x, \ x \geq 0.$$

Before announcing our principal theorem, we recall [55] that the condition $c \geq c_*$ implies that the characteristic equation at the trivial steady state

$$\chi_0(\lambda) := \lambda^2 - c\lambda - 1 + g'(0)e^{-\lambda ch} = 0$$

has exactly two real roots $\lambda_1 = \lambda_1(c) \leq \lambda_2 = \lambda_2(c)$ (counting multiplicity), both of them are positive. Note also that $-\lambda_1(c), \lambda_2(c)$ are increasing functions of c.

The main result of this part is the following

Theorem I.4. *Assume that the initial function w_0 satisfies the hypotheses $(IC1')$, $(IC2')$ and that, for some $A > 0$ and $c > c_*$, it holds*

$$\lim_{x \to -\infty} w_0(s,x)e^{-\lambda_1(c)(x+cs)} = A$$

uniformly on $s \in [-h, 0]$. If, in addition, the birth function g satisfies **(H)**, *then*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \frac{|\phi(x+ct+a) - u(t,x)|}{\eta_{\lambda_1}(x+ct)} = 0, \tag{1.10}$$

where $a = (\lambda_1(c))^{-1} \ln A$ and the front profile ϕ (existing in virtue of the assumption $c > c_*$) is normalised by $\lim_{x\to-\infty} e^{-\lambda_1(c)x}\phi(x) = 1$.

Theorem I.4 allows to answer the velocity selection question for solutions with initial data possessing exponential decay at $-\infty$. Indeed, suppose that, for some $\lambda > 0$, it holds

$$\lim_{x\to-\infty} w_0(s,x)e^{-\lambda x} = A(s) > 0, \text{ uniformly in } s \in [-h, 0]. \tag{1.11}$$

Then define $c(\lambda)$ by the formula $c(\lambda) = \mu/\lambda$, where μ is the unique positive root of the equation

$$\lambda^2 - \mu - 1 + g'(0)e^{-\mu h} = 0.$$

It is easy to see that $c(\lambda) \geq c_\#$, where $c_\# = c_\#(g'(0), h)$ is the so-called critical speed (a uniquely determined value of c for which the characteristic function $\chi_0(\lambda)$ has a double positive zero). Set $\lambda_* := \lambda_1(c_*)$. We claim that

$$c_\lambda := \begin{cases} c(\lambda), & \text{if } \ \lambda < \lambda_*, \\ c_*, & \text{if } \ \lambda \geq \lambda_*, \end{cases}$$

is the speed of propagation selected by solutions with initial data satisfying (1.11). More precisely, the following assertion holds.

Corollary 1. *Assume that the initial function w_0 satisfies the hypotheses $(IC1')$, $(IC2')$ and (1.11). Suppose first that $\lambda > \lambda_*$ and $c_* > c_{\#}$, then*

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \frac{|\phi_*(x+c_*t) - u(t,x)|}{\eta_\nu(x+c_*t)} = 0$$

for each fixed $\nu \in (\lambda_*, \lambda)$. Here ϕ_* denotes the profile of appropriately shifted unique minimal (pushed) front to equation (1.1).

Next, let $\lambda < \lambda_*$ (so that $c(\lambda) = c_\lambda$) and $c_* \geq c_{\#}$. Set

$$a_- := \frac{1}{\lambda} \ln \left[\min_{s\in[-h,0]} A(s)e^{-\mu s} \right] \leq a_+ := \frac{1}{\lambda} \ln \left[\max_{s\in[-h,0]} A(s)e^{-\mu s} \right].$$

Then for every $\epsilon > 0$ there exists $T_1(\epsilon) > 0$ such that

$$(1-\epsilon)\phi_\lambda(x + c_\lambda t + a_-) \leq u(t,x) \leq (1+\epsilon)\phi_\lambda(x + c_\lambda t + a_+), \quad t \geq T_1(\epsilon), \ x \in \mathbb{R}.$$

Here ϕ_λ denotes the profile of the unique wavefront to equation (1.1) propagating with the velocity $c(\lambda)$ and satisfying $\lim_{x\to-\infty} e^{-\lambda x}\phi_\lambda(x) = 1$.

Now, if $\lambda = \lambda_*$ and $c_* > c_{\#}$, then for every $\epsilon > 0$ and positive $\nu < \lambda_*$ there are $T_2(\epsilon) > 0$ and $a' \in \mathbb{R}$ such that

$$(1-\epsilon)\phi_*(x + c_*t + a') \leq u(t,x) \leq (1+\epsilon)\phi_\nu(x + c_\nu t), \quad t \geq T_2(\epsilon), \ x \in \mathbb{R}. \tag{1.12}$$

Furthermore, in such a case, $u(t,x)$ can not converge (al least, uniformly on $\mathbb{R}$) to a wavefront solution of equation (1.1).

Finally, if $L_g = g'(0)$ *(so that* $c_* = c_\#$*) and* $\lambda \geq \lambda_*$*, then for every* $\epsilon > 0$ *there are* $T_3(\epsilon) > 0$ *and* $b' \in \mathbb{R}$ *such that*

$$0 \leq u(t,x) \leq (1+\epsilon)\phi_*(x + c_* t + b'), \quad t \geq T_3(\epsilon), \ x \in \mathbb{R}. \tag{1.13}$$

It is worth to note that there is an important difference between the speed selection results obtained in the non-delayed and delayed cases. Indeed, if $h = 0$ and $\lambda < \lambda_*$ then $a_- = a_+$ and therefore $u(t,x)$ converges to a single wavefront $\phi_\lambda(x + c_\lambda t + a_\pm)$ propagating with the velocity $c_\lambda = \lambda + (g'(0) - 1)/\lambda$. In the delayed case, however, we only can say that $u(t,x)$ evolves between two shifted traveling fronts, both of them moving with the same velocity c_λ. Observe also that, since $\mu = \mu(h)$ is a decreasing function of h, the inclusion of delay in problems modeled by (1.1) slows down the propagation of 'concentrations' having the same initial distribution which satisfies (1.11).

Remark 1. Consider again the final statement of Corollary 1. Under conditions assumed in it (at least when additionally $\lambda > \lambda_*$), it is natural to expect [46] the so-called *convergence in form* of $u(t,x)$ to the minimal wavefront: that is

$$\sup_{x \in \mathbb{R}} |u(t,x) - \phi_*(x + \psi(t))| \to 0, \ \text{as } t \to +\infty,$$

for an appropriate function $\psi(t)$. Then (1.13) implies that the function $\psi(t) - c_* t$ is bounded from above: in other words, in such a case, the concentration $u(t,x)$ should propagate behind the minimal front. A more detailed analysis of this phenomenon for some delayed reaction-difusion models will be given in the forthcoming work by the authors.

Another immediate consequence of Theorem I.4 is the following assertion concerning the global asymptotic stability (without asymptotic phase) of wavefronts:

Corollary 2. *Let g and w_0 satisfy the assumptions* **(H)** *and* $(IC1')$, $(IC2')$. *If*

$$(1.14) \qquad \sup_{s\in[-h,0]} |\phi(\cdot + cs) - w_0(s,\cdot)|_\mu < \infty$$

for some $c > c_*$ and $\mu > \lambda_1(c)$, then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \frac{|\phi(x+ct) - u(t,x)|}{\eta_{\lambda_1}(x+ct)} = 0.$$

Clearly, the statement of Theorem I.4 (or Corollary 2) implies the uniqueness (up to a translation) of non-critical traveling fronts propagating with the same velocity c and having the same order of exponential decay at $-\infty$, cf. e.g. [32, Theorem 1.1], [58, Corollary 4.9]. In any event, the uniqueness of each front (including critical one) to the monotone model (1.1) was established in [55, Theorem 1.2] by means of the Berestycki-Nirenberg method of the sliding solutions. In the case when g is non-monotone, the wave uniqueness was investigated in [1], by applying a suitable L^2-variant of the bootstrap argument suggested by Mallet-Paret in [34]. We recall here that, in the case of a unimodal birth-function g, equation (1.1) can possess non-monotone wavefronts (either slowly oscillating or eventually monotone). This fact was deduced in [11, 20, 54] from the seminal results [33, 34, 35, 36] by Mallet-Paret and Sell.

It is instructive to compare Theorem I.4 and Corollary 2 with the corresponding results from the above mentioned works [37, 38, 39, 40, 58] (restricting them to the particular family of the Mackey-Glass type diffusive equations (1.1)). It is easy to check that Theorem I.4 amplifies Theorem 4.1 from [58] which was proved under more restrictive smoothness and geometric conditions on g and w_0. (Theorem I.5A below also extends the mentioned result by Wang *et al.* for the critical case $c = c_\#$). In particular, the assumptions of [58] contain the inequality $g'(x) \leq g'(0)$, $x \geq 0$, which excludes from consideration the pushed waves, see [55, Subsection 1.2] for more detail. The approach of [58] is a version of the super- and sub-solutions method proposed in [7] and then further developed in [32]. The proofs given in the present paper are also based on the squeezing technique and the Phragmèn-Lindelöf principle for reaction-diffusion equations. Hence, we are also using adequate super- and sub-solutions (which generally are not C^1-smooth and are simpler than those considered in [7, 32, 58]. In particular, the latter fact allows to shorten the proofs).

Another important approach to the wave stability problem in (1.1) is a weighted energy method developed by Mei *et al.* [37, 38, 39, 40]. See also Kyrychko *et al.* [25], Lv and Wang [29], Wu *et al.* [64]. This method is based on rather technical weighted energy estimations and generally requires better properties from g and w_0. For instance, it was assumed in [29, 39] that $g''(x) \leq 0$, $x \geq 0$, and that the weighted initial perturbation $\delta(s,x) = (\phi(x+cs) - w_0(s,x))/\eta_\mu(x)$ belongs to the Sobolev space $H^1(\mathbb{R})$ for some $\mu > \lambda_1$ and for each fixed $s \in [-h, 0]$. It was also assumed in [29, 39] that $\delta : [-h, 0] \to H^1(\mathbb{R})$ is a continuous function that implies immediately the fulfilment of (1.14), in virtue of the corresponding embedding theorem. Therefore Corollary 2 can be also used in such a situation. However, in difference with Corollary 2, the weighted energy method allows to prove the *exponential* stability of non-critical traveling fronts. Consequently, it gives the same convergence rates as the Sattinger functional analytical approach [48] gives in the case of non-delayed version of (1.1). We recall that the latter approach is based on the spectral analysis of equation (1.1) linearised along a wavefront. Thus a certain disadvantage of Theorem I.4 as well as [7, Theorem 2], [32, Theorem 5.1], [58, Theorem 4.1] is that they do not give any estimation of the rate of convergence in (1.10). In this regard, it is a remarkable fact that super- and sub-solutions used in this work are also suitable to provide rather short proofs of the exponential stability [asymptotical stability] of non-critical [respectively, critical] wavefronts in equation (1.1) considered with the birth function g satisfying relatively weak restrictions (**H**) and $L_g = g'(0)$. For example, $L_g = g'(0)$ if g is differentiable on $\mathbb{R}_+$ where $g'(x) \le g'(0)$.

Theorem I.5. *In addition to (**H**), suppose that $L_g = g'(0)$. If the initial function w_0 satisfies the assumptions (IC1'), (IC2'), then the following holds.*

A. If $c \ge c_\#$ and

$$\lim_{z \to -\infty} w_0(s,x)/\phi(x) = 1,$$

uniformly on $s \in [-h, 0]$, then

$$|u(t,\cdot)/\phi(\cdot + ct) - 1|_0 = o(1), \; t \to +\infty. \tag{1.15}$$

B. If $c > c_\#$ and $\lambda \in (\lambda_1(c), \lambda_2(c))$ then

$$\sup_{s \in [-h,0]} |\phi(\cdot + cs) - w_0(s,\cdot)|_\lambda < \infty \tag{1.16}$$

implies that the solution $u(t,x)$ of (1.1), (1.2) satisfies

$$\sup_{x \in \mathbb{R}} \frac{|u(t,x) - \phi(x+ct)|}{\eta_\lambda(x+ct)} \le Ce^{-\gamma t}, \; t \ge 0,$$

for some $C > 0$ and $\gamma > 0$.

To the best of our knowledge, the description of front convergence in the form (1.15) was proposed by Chen and Guo [7]. Clearly, this kind of convergence is equivalent to the weighted convergence expressed by (1.10) (if $c > c_\#$) and it is stronger than the uniform convergence

$$\sup_{x \in \mathbb{R}} |u(t,x) - \phi(x+ct)| \to 0, \; t \to +\infty.$$

The stability results stated in Theorem I.5 have the global character in that sense that none smallness restriction is imposed on the norm (1.16) of perturbation $\phi(x +$

$cs) - w_0(s,x)$. Remarkably, in the case where we do not assume anymore that g is monotone, our approach still allows us to prove the local stability of fronts. Even more than that, we are also able to present some global stability results. In this way, our next main theorem and its corollary can be regarded as a further development of [28, Theorem 2.1] and [64, Theorems 2.4 and 2.6]. Before formulating the corresponding assertions, let us recall that the hypothesis

(UM) Let (**H**) be satisfied except for the monotonicity of g and suppose that $L_g = g'(0)$ and g is bounded on $\mathbb{R}_+$

implies the existence of a unique normalised (at $-\infty$) positive semi-wavewfront $u(t,x) = \phi_c(x+ct)$ to equation (1.1) for each $c \geq c_\#$, see e.g. [1, 19]. We recall here that the definition of a semi-wavewfront is similar to the definition of a wavefront: the only part that is changing is the boundary condition $\phi_c(+\infty) = \kappa$ which should be replaced with $\liminf_{x\to+\infty} \phi_c(x) > 0$.

Theorem I.6. *Assume (**UM**) and let the initial function w_0 satisfy (IC1'). For $c > c_\#$ we take $\lambda \in (\lambda_1(c), \lambda_2(c))$ and for $c = c_\#$ we take $\lambda = \lambda_*$. If we denote $\xi(x,\lambda) = e^{\lambda x}$, then the following holds.*

A. The inequality (1.16) implies that the solution $u(t,x)$ of (1.1), (1.2) converges to the semi-wavefront $\phi_c(x+ct)$: more precisely, there are positive C, γ such that

$$\sup_{x\in\mathbb{R}} \frac{|u(t,x) - \phi(x+ct)|}{\xi(x+ct,\lambda)} \leq Ce^{-\gamma t},\ t \geq 0.$$

B. Let, in addition, $|g'(u)| < 1$ on some interval $[\kappa-\rho, \kappa+\rho]$, $\rho > 0$. If, for some $b \in \mathbb{R}$, the initial function w_0 and the semi-wavefront profile ϕ_c satisfies: $\phi(x-2ch) \in (\kappa-\rho/2, \kappa+\rho/2)$, for $x \geq b$. Then ϕ is actually a wavefront (i.e.

$\phi(+\infty) = \kappa$) and

$$|w_0(s, x + cs) - \phi(x + cs)| \leq qe^{\lambda(x+cs-b)}, \ (s, x) \in \Pi_0,$$

for $q \in (0, \rho/2]$ implies that the solution $u(t, x)$ of (1.1), (1.2) satisfies

$$\sup_{x\in\mathbb{R}} \frac{|u(t, x) - \phi(x + ct)|}{\eta_\lambda(x + ct)} \leq 0.5\rho e^{-\gamma t}, \ t \geq 0. \tag{1.17}$$

where $\gamma \geq 0$ is zero iff $c = c_\#$.

Corollary 3. *Let g satisfy (**UM**) and let g be a unimodal function, with a unique point $x_m \in (0, \kappa)$ of local extremum (maximum). Suppose further that $|g'(x)| < 1$ for all $x \in [g(g(x_m)), g(x_m)]$. Additionally, assume that the initial function w_0 satisfy (IC1') and (IC2') and consider $c > c_\#$, $\lambda \in (\lambda_1(c), \lambda_2(c))$. Then inequality (1.16) implies that the solution $u(t, x)$ of (1.1), (1.2) uniformly converges to the semi-wavefront $\phi(x + ct)$. More precisely, there are positive C, γ such that (1.17) holds.*

Let us illustrate Corollary 3 by considering the well-known diffusive version of the Nicholson's blowflies equation

$$u_t(t, x) \ = \ u_{xx}(t, x) - \delta u(t, x) + pu(t - \tau, x)e^{-u(t-\tau,x)}. \tag{1.18}$$

By realising a re-scaling in space-time, we can transform this equation into the form (1.1) with $g(x) = (p/\delta)xe^{-x}$ and $h = \tau\delta$. In the last decade, the wavefront solutions of equation (1.18) have been investigated by many authors, e.g. see [8, 11, 20, 28, 29, 31, 37, 39, 40, 58, 64]. If the positive parameters p, δ are such that $1 < p/\delta \leq e$, then g satisfies the hypothesis **(H)** with $L_g = g'(0)$ and $\kappa = \ln(p/\delta)$. In such a case, Theorem I.5 guarantees the global stability of all wavefronts, including the minimal one (these wavefronts are necessarily monotone). For the first time, such a global stability result was established by Mei et al. in [39]. Now, if $e < p/\delta < e^2$, the restriction of g on $[0, \kappa]$ is not monotone anymore. Nevertheless, we still have that $L_g = g'(0)$ while the inequality $|g'(x)| < 1$ holds for all $x \in [g(g(x_m)), g(x_m)]$, with $x_m = 1$. Therefore, for each $p/\delta \in (e, e^2)$, Corollary 3 assures the global exponential stability of all non-critical wavefronts to equation (1.18). Note that profiles of these wavefronts are not necessarily monotone and they can oscillate slowly around κ or be non-monotone but eventually monotone at $+\infty$, cf. [11, 20, 28]. Observe also that the upper estimation e^2 for p/δ is optimal [20, 54]. Under the same restriction on p/δ, the local stability of wavefronts to (1.18) was investigated in [28, 64].

CHAPTER II

Asymptotic convergence to pushed wavefronts

2.1 Proof of Theorems I.1 and I.2

Let $u = \phi(x + c_*t)$, $c_* > c_\#$, be a pushed traveling front to equation (1.1). In the sequel, to simplify the notation, we will avoid the subscript $*$ in c_* so that $u = \phi(x + c_*t) = \phi(x + ct)$. As it is usual, we consider the moving coordinate frame (t, z) where $z = x + ct$. Set $w(t, z) = u(t, z - ct)$, then equation (1.1) takes the form

(2.1) $$w_t(t, z) = w_{zz}(t, z) - cw_z(t, z) - w(t, z) + g(w(t - h, z - ch)),$$

(2.2) $$w(s, z) = \tilde{w}_0(s, z) := w_0(s, z - cs),\ (s, z) \in \Pi_0.$$

First, following Fife and McLeod [14, Lemma 4.1] and Rothe [45, Lemma 1], we prove the next assertion.

Lemma 1. *Assume that the hypothesis* (**H**) *is satisfied. Then there exist positive constants* γ, C, q_0^+ *(depending only on* g, ϕ, c, h, λ*) and* $q_0^- = \sigma$ *such that the inequality*

(2.3) $$0 \leq w(s, z) \leq \phi(z) + q\eta(z), \quad (s, z) \in \Pi_0,$$

with $q \in (0, q_0^+]$ *implies*

(2.4) $$0 \leq w(t, z) \leq \phi(z + Cq) + qe^{-\gamma t}\eta(z), \quad z \in \mathbb{R}, \quad t \geq -h.$$

Similarly, the inequality

(2.5) $$\phi(z + Cq) - q\eta(z) \leq w(s, z) \leq \kappa, \quad (s, z) \in \Pi_0,$$

with $q \in (0, q_0^-]$ *implies*

$$\phi(z) - qe^{-\gamma t}\eta(z) \le w(t,z) \le \kappa, \quad z \in \mathbb{R}, \quad t \ge -h. \tag{2.6}$$

Proof. For the convenience of the reader, the proof is divided into five steps. Recall that the positive numbers δ_0, σ are defined in (**H**) and (**IC**), respectively.

Step I. We claim that given $\sigma \in (0, \kappa)$, there are positive $\delta_1^* < \delta_0,\ \gamma_1^* < \lambda c$ such that

$$g(u) - g(u - qe^{\gamma h}) \le q(1 - 2\gamma), \text{ for all } (u, q) \in [\kappa - \delta_1^*, \kappa + \delta_1^*] \times [0, \sigma],\ \gamma \in [0, \gamma_1^*].$$

Indeed, it suffices to note that, given $\sigma \in (0, \kappa)$, the continuous function

$$G(u, q, \gamma) := \begin{cases} 1 + (g(u - e^{\gamma h}q) - g(u))/q, & (u, q, \gamma) \in [\kappa - \delta_1^*, \kappa + \delta_1^*] \times (0, \sigma] \times [0, \gamma_1^*], \\ 1 - e^{\gamma h} g'(u), & u \in [\kappa - \delta_1^*, \kappa + \delta_1^*],\ q = 0,\ \gamma \in [0, \gamma_1^*], \end{cases}$$

satisfies $G(\kappa, q, 0) > 2\gamma_1^*,\ q \in [0, \sigma]$, for sufficiently small γ_1^*, δ_1^* (recall that $g'(\kappa) < 1$). Thus $G(u, q, \gamma) > 2\gamma$ for all $(u, q) \in [\kappa - \delta_1^*, \kappa + \delta_1^*] \times [0, \sigma],\ \gamma \in [0, \gamma_1^*]$ if γ_1^*, δ_1^* are sufficiently small.

Step II. As in [14, 45], we have to construct appropriate super- and sub-solutions. Consider the nonlinear operator $\mathcal{N}$ defined as

$$\mathcal{N}w(t, z) := w_t(t, z) - w_{zz}(t, z) + cw_z(t, z) + w(t, z) - g(w(t - h, z - ch)).$$

By definition, continuous function $w_+ : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is called a super-solution for (2.1), if, for some $z_* \in \mathbb{R}$, this function is $C^{1,2}$-smooth in the domains $\mathbb{R}_+ \times (-\infty, z_*]$ and $\mathbb{R}_+ \times [z_*, +\infty)$ and

$$\mathcal{N}w_+(t, z) \ge 0 \text{ for } t > 0,\ z \ne z_*, \tag{2.7}$$

while

$$(w_+)_z(t, z_*-) \ge (w_+)_z(t, z_*+) \text{ for } t > 0. \tag{2.8}$$

Sub-solutions w_- are defined analogously, with the inequalities "$\geq$" reversed in (2.7) and (2.8). We will look for super- and sub-solutions of the form

$$w_+(t,z) := \phi(z+\epsilon(t)) + qe^{-\gamma t}\eta(z), \quad w_-(t,z) := \phi(z-\epsilon_1(t)) - qe^{-\gamma t}\eta(z),$$

where, for appropriate positive parameters α, γ (to be fixed later and depending only on g, ϕ, c, h, λ), increasing $\epsilon(t)$, $\epsilon_1(t)$ are defined by

$$\epsilon(t) := \frac{\alpha q}{\gamma}(e^{\gamma h} - e^{-\gamma t}) > 0, \quad \epsilon_1(t) := -\frac{\alpha q}{\gamma}e^{-\gamma t} < 0, \quad t > -h.$$

Note that the smoothness conditions and the second inequality in (3.3) with $z_* = 0$ are obviously fulfilled because of

$$\frac{\partial w_+(t,0+)}{\partial z} - \frac{\partial w_+(t,0-)}{\partial z} = -q\lambda e^{-\gamma t} < 0, \quad \frac{\partial w_-(t,0+)}{\partial z} - \frac{\partial w_-(t,0-)}{\partial z} = q\lambda e^{-\gamma t} > 0,$$

so that we have to check (2.7) only. Since g, ϕ, ϵ are strictly increasing, we have, for $z \neq 0$, that

$$\begin{aligned}\mathcal{N}w_+(t,z) &:= \epsilon'(t)\phi'(z+\epsilon(t)) - \gamma qe^{-\gamma t}\eta(z) - \phi''(z+\epsilon(t)) - qe^{-\gamma t}\eta''(z) + \\ &c\phi'(z+\epsilon(t)) + cqe^{-\gamma t}\eta'(z) + \phi(z+\epsilon(t)) + qe^{-\gamma t}\eta(z) - g(w_+(t-h, z-ch)) \\ &\geq \alpha qe^{-\gamma t}\phi'(z+\epsilon(t)) - \gamma qe^{-\gamma t}\eta(z) + cqe^{-\gamma t}\eta'(z) + qe^{-\gamma t}\eta(z) - qe^{-\gamma t}\eta''(z) \\ &\quad + g(\phi(z-ch+\epsilon(t))) - g(\phi(z-ch+\epsilon(t)) + qe^{-\gamma(t-h)}\eta(z-ch));\end{aligned}$$

$$\begin{aligned}\mathcal{N}w_-(t,z) &:= -\epsilon_1'(t)\phi'(z-\epsilon_1(t)) + \gamma qe^{-\gamma t}\eta(z) - \phi''(z-\epsilon_1(t)) + qe^{-\gamma t}\eta''(z) + \\ &c\phi'(z-\epsilon_1(t)) - cqe^{-\gamma t}\eta'(z) + \phi(z-\epsilon_1(t)) - qe^{-\gamma t}\eta(z) - g(w_-(t-h, z-ch)) \\ &\leq -\alpha qe^{-\gamma t}\phi'(z-\epsilon_1(t)) + \gamma qe^{-\gamma t}\eta(z) - cqe^{-\gamma t}\eta'(z) - qe^{-\gamma t}\eta(z) + qe^{-\gamma t}\eta''(z) + \\ &\quad g(\phi(z-ch-\epsilon_1(t))) - g(\phi(z-ch-\epsilon_1(t)) - qe^{-\gamma(t-h)}\eta(z-ch)).\end{aligned}$$

Since $\lambda \in (\lambda_1(c), \lambda_2(c))$ and $g'(0) > 1$, we can choose sufficiently small $\gamma \in (0, \gamma_1^*)$ and $\delta \in (0, \kappa/2) \cap (0, \delta_1^*) \cap (0, \sigma)$, such that, for all $\bar{s} < \delta$ it holds

$$-\lambda^2 + c\lambda + 1 - \gamma - g'(\bar{s})e^{-\lambda ch + \gamma h} > 0. \tag{2.9}$$

In addition, we can take δ such that the unique real roots $z_0 < z_1 < z_2$ of the equations

$$\phi(z_0) = \delta/4, \quad \phi(z_1) + 0.25\delta\eta(z_1) = \delta/2; \quad \phi(z_2) = \kappa - \delta/2,$$

are such that $z_1 < -ch < 0 < z_2$. From now on, we will fix $\alpha, q_0^{\pm}$ defined by

$$q_0^+ = \delta e^{-\gamma h}/2, \quad q_0^- = \sigma, \quad \alpha = (\gamma + e^{\gamma h} L_g)/\beta, \quad \text{with } \beta := \min_{z \in [z_0, z_2 + ch]} \phi'(z).$$

We observe that $\alpha, q_0^{\pm}$ and γ depends only on $g, \phi, c, h, \lambda, \sigma$.

Step III. We claim that $\mathcal{N} w_+(t,z) \geq 0$ for all $z \neq 0,\ t \geq 0$ and $q \leq q_0^+ = \delta e^{-\gamma h}/2$.

Indeed, suppose first that $z - ch + \epsilon(t) \leq z_1$, then $z \leq z_1 + ch - \epsilon(t) < -\epsilon(t) < 0$ and

$$\phi(z - ch + \epsilon(t)) < \delta/2,\ \phi(z - ch + \epsilon(t)) + q e^{-\gamma(t-h)} \eta(z - ch) < \delta.$$

As a consequence, we can invoke the mean value theorem and and (2.9) to conclude that, for some $\bar{s} \in (0, \delta)$,

$$g(\phi(z - ch + \epsilon(t))) - g(w_+(t-h, z - ch)) = -q e^{-\gamma(t-h)} \eta(z - ch) g'(\bar{s}),$$

$$\mathcal{N} w_+(t,z) \geq q e^{-\gamma t} [-\gamma \eta(z) + c\eta'(z) + \eta(z) - \eta''(z) - e^{\gamma h} \eta(z - ch) g'(\bar{s})]$$

$$= q e^{-\gamma t + \lambda z} \left(1 - \gamma + c\lambda - \lambda^2 - e^{\gamma h - \lambda ch} g'(\bar{s})\right) > 0.$$

Similarly, if $z - ch + \epsilon(t) \geq z_2$, then we have that

$$\kappa + \delta/2 \geq \phi(z - ch + \epsilon(t)) + q e^{-\gamma(t-h)} \eta(z - ch) \geq \phi(z - ch + \epsilon(t)) \geq \kappa - \delta/2.$$

Therefore, due to Step I and (2.9), for all $t \geq 0$,

$$g(\phi(z - ch + \epsilon(t))) - g(\phi(z - ch + \epsilon(t)) + q e^{-\gamma(t-h)} \eta(z - ch)) \geq -q e^{-\gamma t} \eta(z - ch)(1 - 2\gamma),$$

$$\mathcal{N} w_+(t,z) \geq q e^{-\gamma t} \left([1-\gamma]\eta(z) + c\eta'(z) - \eta''(z) - (1-2\gamma)\eta(z - ch)\right) \geq$$

$$q e^{-\gamma t} \left\{ \begin{array}{ll} e^{\lambda z}[1 - \gamma + c\lambda - \lambda^2 - e^{-\lambda ch}(1 - 2\gamma)], & z < 0 \\ \gamma, & z > 0 \end{array} \right\} > 0.$$

Finally, if $z_1 < z - ch + \epsilon(t) < z_2$, we find that

$$\phi(z - ch + \epsilon(t)) < \kappa - \delta/2, \quad \phi(z - ch + \epsilon(t)) + 0.25\delta\eta(z - ch + \epsilon(t)) > \delta/2$$

so that $\phi(z-ch+\epsilon(t)) > \delta/2 - 0.25\delta\eta(z-ch+\epsilon(t)) > \delta/4$, and $z+\epsilon(t) \in [z_0, z_2+ch]$. Obviously,

$$|g(\phi(z-ch+\epsilon(t))) - g(\phi(z-ch+\epsilon(t)) + qe^{-\gamma(t-h)}\eta(z-ch))| \le L_g q e^{-\gamma(t-h)}\eta(z-ch).$$

Therefore, since $\eta(z) + c\eta'(z) - \eta''(z) > 0$ for $z \neq 0$ and $\eta(z) \in (0,1]$, we get

$$\mathcal{N}w_+(t,z) \ge qe^{-\gamma t}\left\{\alpha\beta + \eta(z) + c\eta'(z) - \eta''(z) - \gamma - e^{\gamma h}L_g\right\} > 0$$

for all $t \ge 0$. Hence, there exist some constants $\alpha, \gamma, q_0^+ > 0$, depending only on the wavefront profile ϕ, the nonlinearity g and c, h, λ such that, for any choice of $q \in (0, q_0^+)$ it holds $\mathcal{N}w_+(t,z) > 0$ for all $t \ge 0$ and $z \neq 0$. This proves the first inequality in (3.3).

Step IV. We claim that $\mathcal{N}w_-(t,z) \le 0$ for all $z \neq 0,\ t \ge 0$ and $q \le q_0^- = \sigma$. Indeed, suppose first that $z - ch - \epsilon_1(t) \le z_1$, then $z \le z_1 + \epsilon_1(t) + ch < 0$ and

$$\phi(z - ch - \epsilon_1(t)) - qe^{-\gamma(t-h)}\eta(z - ch) < \phi(z - ch - \epsilon_1(t)) < \delta/2.$$

As a consequence, the mean value theorem yields that for some $\bar{s} < \delta$,

$$g(\phi(z - ch - \epsilon_1(t))) - g(w_-(t-h, z-ch)) = qe^{-\gamma(t-h)}\eta(z-ch)g'(\bar{s}),$$

$$\mathcal{N}w_-(t,z) \le qe^{-\gamma t}[\gamma\eta(z) - cqe^{-\gamma t}\eta'(z) - \eta(z) + \eta''(z) + e^{\gamma h}\eta(z-ch)g'(\bar{s})]$$

$$= -qe^{-\gamma t}e^{\lambda z}[1 - \gamma + c\lambda - \lambda^2 - e^{\gamma h - \lambda ch}g'(\bar{s})] < 0.$$

Similarly, if $z - ch - \epsilon_1(t) \ge z_2$, then we have that $\phi(z - ch - \epsilon_1(t)) \ge \kappa - \delta/2$ and therefore

$$g(\phi(z-ch-\epsilon_1(t))) - g(\phi(z-ch-\epsilon_1(t)) - qe^{-\gamma(t-h)}\eta(z-ch)) \le (1-2\gamma)qe^{-\gamma t}\eta(z-ch)$$

for all $t \geq 0,\ q \in [0, \sigma]$. In consequence,

$$\mathcal{N}w_-(t,z) \leq -qe^{-\gamma t}\left([1-\gamma]\eta(z) + c\eta'(z) - \eta''(z) - (1-2\gamma)\eta(z-ch)\right) \leq$$
$$-qe^{-\gamma t}\left\{\begin{array}{ll} e^{\lambda z}[1-\gamma+c\lambda-\lambda^2-e^{-\lambda ch}(1-2\gamma)], & z<0 \\ \gamma, & z>0 \end{array}\right\} < 0.$$

Finally, if $z_1 < z - ch - \epsilon_1(t) < z_2$, we find that

$$\phi(z-ch-\epsilon_1(t)) < \kappa - \delta/2, \quad \phi(z-ch-\epsilon_1(t)) + 0.25\delta\eta(z-ch-\epsilon_1(t)) > \delta/2$$

so that $\phi(z-ch-\epsilon_1(t)) > \delta/2 - 0.25\delta\eta(z-ch-\epsilon_1(t)) > \delta/4$, and $z-\epsilon_1(t) \in [z_0, z_2+ch]$. Obviously,

$$|g(\phi(z-ch-\epsilon_1(t))) - g(\phi(z-ch-\epsilon_1(t)) - qe^{-\gamma(t-h)}\eta(z-ch))| \leq L_g qe^{-\gamma(t-h)}\eta(z-ch).$$

Therefore, since $\eta(z) + c\eta'(z) - \eta''(z) > 0$ for $z \neq 0$ and $\eta(z) \in (0,1]$, we get

$$\begin{aligned} \mathcal{N}w_-(t,z) &\leq -qe^{-\gamma t}\{\alpha\beta + \eta(z) + c\eta'(z) - \eta''(z) - \gamma - e^{\gamma h}L_g\} \\ &< -qe^{-\gamma t}(\gamma + L_g) < 0 \end{aligned}$$

for $t \geq 0$.

Step V. In view of (12) and the monotonicity properties of g, we have that

$$g(w_+(t-h, z-ch)) - g(w(t-h, z-ch)) \geq 0, \quad t \in [0,h],\ z \in \mathbb{R}.$$

Therefore,for $(t,z) \in (0,h] \times (\mathbb{R} \setminus \{0\})$, the difference $\delta(t,z) := w(t,z) - w_+(t,z)$ satisfies the relations

$$\delta(0,z) \leq 0,\ |\delta(t,z)| \leq \kappa + q_0^+, \quad \delta_{zz}(t,z) - \delta_t(t,z) - c\delta_z(t,z) - \delta(t,z) =$$
$$\mathcal{N}w_+(t,z) - \mathcal{N}w(t,z) + g(w_+(t-h,z-ch)) - g(w(t-h,z-ch)) =$$
$$\mathcal{N}w_+(t,z) + g(w_+(t-h,z-ch)) - g(w(t-h,z-ch)) \geq 0,$$
$$\frac{\partial\delta(t,0+)}{\partial z} - \frac{\partial\delta(t,0-)}{\partial z} = q\lambda e^{-\gamma t} > 0. \tag{2.10}$$

We claim that $\delta(t,z) \le 0$ for all $t \in [0,h],\ z \in \mathbb{R}$. Indeed, otherwise there exists $r_0 > 0$ such that $\delta(t,z)$ restricted to any rectangle $\Pi_r = [-r,r] \times [0,h]$ with $r > r_0$, reaches its maximal positive value $M > 0$ at at some point $(t',z') \in \Pi_r$.

We claim that (t',z') belongs to the parabolic boundary $\partial\Pi_r$ of Π_r. Indeed, suppose on the contrary, that $\delta(t,z)$ reaches its maximal positive value at some point (t',z') of $\Pi_r \setminus \partial\Pi_r$. Then clearly $z' \neq 0$ because of (2.8). Suppose, for instance that $z' > 0$. Then $\delta(t,z)$ considered on the subrectangle $\Pi = [0,r] \times [0,h]$ reaches its maximal positive value M at the point $(t',z') \in \Pi \setminus \partial\Pi$. Then the classical results [43, Chapter 3, Theorems 5,7] shows that $\delta(t,z) \equiv M > 0$ in Π, a contradiction.

Hence, the usual maximum principle holds for each $\Pi_r,\ r \ge r_0$, so that we can appeal to the proof of the Phragmèn-Lindelöf principle from [43] (see Theorem 10 in Chapter 3 of this book), in order to conclude that $\delta(t,z) \le 0$ for all $t \in [0,h],\ z \in \mathbb{R}$.

But then we can again repeat the above argument on the intervals $[h,2h],\ [2h,3h],\dots$ establishing that the inequality

$$0 \le w(s,z) \le \phi(z+\epsilon(s)) + qe^{-\gamma s}\eta(z), \quad z \in \mathbb{R},$$

actually holds for all $s \ge -h$. Since $\epsilon(t)$ increases on $\mathbb{R}$, this proves (2) with $C = \epsilon(\infty) = \alpha e^{\gamma h}/\gamma$.

Since the same method applied (with $C = \alpha e^{\gamma h}/\gamma$ in (2)) to the difference $\delta_-(t,z) := w_-(t,z) - w(t,z)$ leads to

$$\phi(z) - qe^{-\gamma s}\eta(z) < \phi(z-\epsilon_1(s)) - qe^{-\gamma s}\eta(z) \le w(t,z) \le \kappa, \quad t \ge -h,\ z \in \mathbb{R},$$

the proof of the lemma is completed. □

Remark 2. It is worthwhile to note that the constants $\gamma, C, q_0^{\pm}$ depend only on the form of ϕ in the sense that they will not change if we replace $\phi(z)$ with a shifted profile $\phi(z+b),\ b \in \mathbb{R}$, in the statement of Lema 1.

Due to Remark 2, the inequalities (2), (2) can be presented in the form similar to (12), (2):

Corollary 4. *Assume that the hypothesis* (**H**) *is satisfied. Then the inequality*

$$\phi(z) - q\eta(z) \le w(s,z) \le \kappa, \quad (s,z) \in \Pi_0,$$

with $q \in (0, \sigma]$ *implies*

$$\phi(z - Cq) - qe^{-\gamma t}\eta(z) \le w(t,z) \le \kappa, \quad z \in \mathbb{R}, \quad t \ge -h.$$

Proof. By Remark 2, the statements of Lema 1 will not change if we replace $\phi(z)$ with a shifted profile $\phi(z+b),\ b \in \mathbb{R}$. Taking $b = -Cq$, we complete the proof of Corollary 4. □

As an immediate consequence of Lemma 1 and Corollary 4, we obtain the stability of the wavefront solution $u(t,x) = \phi(x+ct)$ with respect to the norm $|\cdot|_\lambda$:

Corollary 5. *For every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $|\phi(\cdot + s_0) - w(s,\cdot)|_\lambda < \delta$, $s \in [-h, 0]$, *implies that* $|\phi(\cdot + s_0) - w(t,\cdot)|_\lambda < \epsilon$ *for all* $t \ge 0$.

Proof. Without loss of generality, we can assume that $s_0 = 0$. From Theorem 1.4 and Proposition 2 from [55], we know that $\phi'(z) = O(e^{\lambda_2 z})$ at $-\infty$. This implies that $|\phi'(z)| \le K\min\{1,\ e^{\lambda_2 z}\}, z \in \mathbb{R}$, for some positive K. In this way, for each fixed $p \in \mathbb{R}$,

$$0 < \phi'(z+p) \le K\min\{1, e^{\lambda_2(z+p)}\} \le Ke^{\lambda_2|p|}\min\{1, e^{\lambda_2 z}\},\ z \in \mathbb{R}.$$

Fix $\epsilon > 0$ and consider $\delta \in (0, q_0^+) \cap (0, \epsilon/(1+K_1))$, where $K_1 = CKe^{\lambda_2 C q_0^+}$. Next, assume that $|\phi(\cdot) - w(s,\cdot)|_\lambda < \delta,\ s \in [-h, 0]$. This yields that

$$\phi(z) - \delta\eta(z) < w(s,z) < \phi(z) + \delta\eta(z),\ (s,z) \in \Pi_0,$$

and therefore, due to Lemma 1 and Corollary 4,

$$\phi(z - C\delta) - \delta\eta(z) < w(t,z) < \phi(z + C\delta) + \delta\eta(z), \ t \geq 0, \ z \in \mathbb{R}.$$

Now, for some $\hat{s} \in (0, C\delta)$, it holds

$$\phi(z + C\delta) = \phi(z) + \phi(z + C\delta) - \phi(z) = \phi(z) + C\delta\phi'(z + \hat{s})$$

$$\leq \phi(z) + CKe^{\lambda_2 C q_0^+}\delta \min\{1, e^{\lambda_2 z}\} \leq \phi(z) + K_1\delta\eta(z). \tag{2.11}$$

After establishing a similar lower bound for $\phi(z - C\delta)$, we get

$$\phi(z) - (K_1 + 1)\delta\eta(z) < w(t,z) < \phi(z) + (K_1 + 1)\delta\eta(z), \ t \geq 0, \ z \in \mathbb{R},$$

that is, $|\phi(\cdot) - w(t,\cdot)|_\lambda < \delta(K_1 + 1) = \epsilon, \ t \geq 0.$ □

In addition, Lemma 1 yields the following useful result

Corollary 6. *Assume that* $w_0(s,x)$ *satisfies* (**IC**). *Then there exist positive* γ, ζ_1 *such that for* $(t,z) \in [-h,\infty) \times \mathbb{R}$ *we have*

$$\phi(z - \zeta_1) - \sigma e^{-\gamma t}\eta(z) \leq w(t,z) \leq \phi(z + \zeta_1) + q_0^+ e^{-\gamma t}\eta(z + \zeta_1). \tag{2.12}$$

Proof. First, we will show that inequality (12) holds for $w_0(s, z - \zeta_0)$ if we take sufficiently large ζ_0. Indeed, let z' be such that $\phi(z') + q_0^+\eta(z') = \kappa$ and define ζ_0 from

$$Ae^{-\mu\zeta_0} = q_0^+ \min\{e^{-\mu z'}, e^{(\lambda-\mu)z'}\}.$$

Then, for all $z \geq z', \ s \in [-h, 0]$, it holds that $w(s, z - \zeta_0) \leq 1 \leq \phi(z) + q_0^+\eta(z)$. Furthermore, because of the assumption $(IC2)$ and the inequality $\lambda < \mu$, we have, for all $z \leq z', \ s \in [-h, 0]$, that $w(s, z - \zeta_0) \leq Ae^{\mu(z-\zeta_0)} =$

$$q_0^+ \min\{e^{-\mu z'}, e^{(\lambda-\mu)z'}\}e^{\mu z} \leq q_0^+ \min\{e^{\mu(z-z')}, e^{\lambda z}\} \leq q_0^+\eta(z) < \phi(z) + q_0^+\eta(z).$$

Therefore, due to (2),

$$0 \le w(t, z-\zeta_0) \le \phi(z + Cq_0^+) + q_0^+ e^{-\gamma t}\eta(z), \quad z \in \mathbb{R}, \quad t \ge -h.$$

Hence, setting $\zeta_1 = \zeta_0 + Cq_0^+$ and using the translation invariance of equation (1.1), we obtain the second inequality in (2.12).

Similarly, there exists z'' such that

$$\phi(z+z'') - q_0^- \eta(z) \le 0 \le w(s, z+B) \le \kappa, \quad z \le 0, \ s \in [-h, 0];$$

$$\phi(z+z'') - q_0^- \eta(z) \le \kappa - \sigma \le w(s, z+B) \le \kappa, \quad z \ge 0, \ s \in [-h, 0].$$

Hence, by (2) and Remark 2, we obtain

$$\phi(z + z'' - Cq_0^-) - q_0^- e^{-\gamma t}\eta(z) \le w(s, z+B) \le \kappa, \quad z \in \mathbb{R}, \quad t \ge -h.$$

As a consequence, the both inequalities in (2.12) hold if we take $\zeta_1 = \zeta_0 + C(q_0^+ + q_0^-) + |B - z''|$. □

Remark 3. Observe that the hypothesis $(IC3)$ was not used to prove the right-hand side inequality in (2.12).

Next, it should be noted that the variable shift $\epsilon(t)$ in $w_+(t,z)$ was needed only to assure the inequality $\mathcal{N}w_+(t,z) \ge 0$ on the finite interval $z - ch + \epsilon(t) \in [z_1, z_2]$, cf. Step III. This observation suggests the following important modification of Lemma 1 (where we will take the same constants $\delta, \gamma > 0$ which were defined in Step II of the proof of Lemma 1):

Lemma 2. *Let $w(t,z)$ be a solution of (2.1), (2.2) with $\tilde{w}_0(s,z) \in [0,\kappa]$. Take $\delta > 0$ as in (2.9) and let $R > ch$ be such that*

$$0 \le w(t,z), \ \phi(z) \le \delta, \ \textit{if } z \le -R + ch, \ t \ge -h, \ \textit{and}$$

$$|w(t,z) - \kappa|, \ |\phi(z) - \kappa| < \delta \ \ \textit{if } z \ge R - ch, \ t \ge -h.$$

Furthermore, suppose that $w(s,z) \le \phi(z) + \delta\eta(z)$ *for all* $(s,z) \in \Pi_0$ *and* $w(t,z) \le \phi(z)$ *for all* $(t,z) \in \mathbb{R}_+ \times [-R-ch, R+ch]$. *Then* $w(t,z) \le \phi(z) + \delta\eta(z)e^{-\gamma t}$ *for all* $z \in \mathbb{R},\ t \ge 0$.

Proof. Set $\rho(t,z) = w(t,z) - \phi(z)$, then, for some $\xi(t,z)$ lying between points $w(t-h, z-ch)$ and $\phi(z-ch)$, if $(t,z) \in [0,\infty) \times \mathbb{R}$ we obtain

$$\rho_t(t,z) = \rho_{zz}(t,z) - c\rho_z(t,z) - \rho(t,z) + g'(\xi(t,z))\rho(t-h, z-ch), \quad z \in \mathbb{R},\ t \ge 0.$$

Since $\xi(t,z) \in [0,\delta]$ for $z \le -R,\ t \ge 0$, and $\kappa - \xi(t,z) \in [0,\delta]$ for $z \ge R,\ t \ge 0$, we find that $r(t,z) := \delta\eta(z)e^{-\gamma t}$, for $|z| \ge R$ and $t > 0$, satisfies

$$r_t(t,z) - r_{zz}(t,z) + cr_z(t,z) + r(t,z) - g'(\xi(z,t))r(t-h, z-ch) =$$
$$\delta e^{-\gamma t}\left((1-\gamma)\eta(z) - \eta''(z) + c\eta'(z) - e^{\gamma h}g'(\xi(t,z))\eta(z-ch)\right) > 0.$$

In addition, by our assumptions, the piece-wise smooth function $\delta(t,z) := w(t,z) - (\phi(z) + r(t,z))$ satisfies the inequalities $\delta(t, \pm R) \le 0,\ |\delta(t,z)| \le 2\kappa + \delta,\ t \ge 0, z \in \mathbb{R}$; $\delta(s,z) \le 0,\ s \in [-h,0],\ z \in \mathbb{R}$. In consequence,

$$\delta_{zz}(t,z) - \delta_t(t,z) - c\delta_z(t,z) - \delta(t,z) > -g'(\xi(t,z))\delta(t-h, z-ch) \ge 0,$$

for all $t \in [0,h],\ |z| \ge R$. By the Phragmèn-Lindelöf principle [43], we conclude that $\delta(t,z) \le 0$ for all $t \in [0,h],\ |z| \ge R$. Since we also have assumed that $w(t,z) \le \phi(z)$ for all $(t,z) \in \mathbb{R}_+ \times [-R-ch, R+ch]$, we obtain that $\delta(t,z) \le 0$ for all $t \in [0,h],\ z \in \mathbb{R}$. Finally, repeating the above arguments on the intervals $[h, 2h],\ [2h, 3h], \ldots,$ we complete the proof of the lemma. $\square$

Finally, before starting with the proof of Theorems I.1 and I.2, we will establish the following compactness result.

Lemma 3. *Assume that continuous function* $w : [-h, +\infty) \times \mathbb{R} \to [0, \kappa]$ *is a classical solution, for* $t > 0$*, of equation (2.1) and that* $t_j \to +\infty$*. Then there exists a subsequence* $\{t_{j_k}\} \subset \{t_j\}$ *such that* $w(t_j + s, z)$ *converges, uniformly on each rectangle* $[-h, 0] \times [-m, m]$*,* $m \in \mathbb{N}$*, to the restriction* $w_*(s, z)$*,* $(s, z) \in \Pi_0$*, of some entire solution* $w_* : \mathbb{R}^2 \to [0, \kappa]$ *of equation (2.1).*

Proof. First, we observe that, for each fixed $t > h$, function $g(w(t - h, z - ch))$ is locally Lipschitz continuous in $z \in \mathbb{R}$ and therefore w, w_z, w_{zz} are Hölder continuous in $(h, +\infty) \times \mathbb{R}$, cf. [30, Theorem 1]. Next, fix an arbitrary positive $T > 2h + 2$ and $m \in \mathbb{N}$ and consider, for $t_j > T + 2h$, solutions $w_j(t, z) = w(t_j + t, z)$, $(t, z) \in D_+ := [-T, T] \times [-m - 1, m + 1 + ch]$, of the equation

$$w_t(t, z) = w_{zz}(t, z) - cw_z - w(t, z) + g_j(t, z),$$

where $g_j(t, z) := g(w_j(t - h, z - ch))$. We claim that, for each $\alpha \in (0, 1)$, there exists a positive K depending only on m, T, α such that the Hölder norms

$$|g_j|_\alpha^D = \sup_{(t,z)\in D} |g_j(t, z)| + \sup_{(t,z)\neq(s,x)\in D} \frac{|g_j(t, z) - g_j(s, y)|}{(|x - z|^2 + |t - s|)^{\alpha/2}}$$

are uniformly bounded in $D := [-T + 1 + h, T] \times [-m, m]$ by K (i.e. $|g_j|_\alpha^D \leq K$ for all j. Observe that $|g_j|_\alpha^{D_+}$ is finite due to [30, Theorem 1]). In fact, since g satisfies the Lipschitz condition on $[0, \kappa]$, it suffices to establish the uniform boundedness of $|w_j|_\alpha^{D_1}$ in a bigger domain $D_1 := [-T + 1, T] \times [-m, m + ch] \subset D_+$. Obviously, w_j solves in D_+ the initial-boundary value problem $w = w_j|_{\partial D_+}$ where $w_j|_{\partial D_+}$ denotes the restriction of w_j on the parabolic boundary $\partial D_+ := \{-T\} \times [-m-1, m+1+ch] \cup [-T, T] \times \{-m-1, m+1+ch\}$ of D_+. Let $\rho : [-T, T] \to [0, 1]$ be some nondecreasing smooth function such that $\rho([-T, -T + 0.25]) = 0$, $\rho([-T + 0.75, T]) = 1$. Then $w_j = w_{j,1} + w_{j,2}$ where $w_{j,1}$ is the solution of the initial-boundary value problem

$w = 0|_{\partial D_+}$ for the equation

$$w_t(t,z) = w_{zz}(t,z) - cw_z - w(t,z) + \rho(t)g_j(t,z),$$

and $w_{j,2}$ solves the initial-boundary value problem $w = w_j|_{\partial D_+}$ for the equation

$$w_t(t,z) = w_{zz}(t,z) - cw_z - w(t,z) + (1-\rho(t))g_j(t,z).$$

Next, since $|g_j(t,z)| \leq \kappa$ for all $(t,z) \in D_+, j \in \mathbb{N}$, a priori estimate (of the type $1+\delta$) established in [15, Theorem 4, Chapter 7] guarantees that $|w_{j,1}|_\alpha^{D_+} \leq K_1, j \in \mathbb{N}$, where K_1 depends only on m, T, α. As consequence, since $\sup_{D_+} |w_j|,\ j \in \mathbb{N}$, are uniformly bounded by κ, we deduce that $\sup_{D_+} |w_{j,2}| = \sup_{D_+} |w_j - w_{j,1}|,\ j \in \mathbb{N}$, are also uniformy bounded. In addition, $(1-\rho(t))g_j(t,z) = 0$ in $[-T+0.75, T] \times [-m-1, m+1+ch]$, so that we can invoke the interior Schauder estimates (see, e.g, [15, Theorem 5, Chapter 3]) in order to deduce that $|w_{j,1}|_\alpha^{D_1} \leq K_2, j \in \mathbb{N}$, where $K_2 > 0$ depends only on α and K_1. Hence, $|w_j|_\alpha^{D_1} \leq K_1 + K_2, j \in \mathbb{N}$, and therefore $|g_j|_\alpha^D \leq K := L_g(K_1 + K_2)$ for all j.

Applying now Theorem 15 from [15, Chapter 3], we conclude that there exists a subsequence $\{t_{j_k}\} \subset \{t_j\}$ such that $w_{j_k}(t,z)$ converges, uniformly on $[-T+2+h, T-1] \times [-m+1, m-1]$, to the classical solution $w_{T,m} : [-T+2+2h, T] \times [-m+1, m-1] \to [0,\kappa]$ of equation (2.1). Finally, considering $m, T \to +\infty$ and applying a standard diagonal argument, we can assume that $w_{j_k}(t,z)$ converges, uniformly on compact subsets of $\mathbb{R}^2$ to an entire classical solution $w_* : \mathbb{R}^2 \to [0,\kappa]$ of the functional differential equation (2.1). Observe that the arguments used to estimate $|w_j|_\alpha^{D_1}$ can be also applied without changes to w_* so that $|w_*|_\alpha^{D_1} \leq K_1 + K_2$ with the same K_1, K_2. □

Remark 4. Due to Lemma 3, we can define ω-limit set $\omega(w_0)$ which consists from the restrictions $w_*(s,z)$, $(s,z) \in \Pi_0$, of all possible entire limit solutions $w_* =$

$\lim_{k\to+\infty} w_{j_k}$ to (2.1) (which are obtained by considering all possible sequences $\{t_j\}$ converging to $+\infty$ in this lemma). Since each w_* is an entire solution, the set $\omega(w_0)$ is invariant. Furthermore, since $|w_*|_\alpha^D \leq K_1 + K_2$ where K_1, K_2 depend only on D and α, the set $\omega(w_0)$ is pre-compact with respect to the topology of the uniform convergence on bounded subsets of Π_0. Actually $\omega(w_0)$ is compact in the mentioned topology since each element of $\omega(w_0)$ can uniformly (on bounded sets) approximated by w_j.

Theorem II.1. *Assume that $u = \phi(x + c_*t)$, $c_* > c_\#$, is a pushed traveling front to equation (1.1). If initial function w_0 satisfies all conditions* (**IC**) *then, for some $z_0 \in \mathbb{R}$, the classical solution $w = w(t,z)$ of the initial value problem (2.1), (2.2) asymptotically converges to a shifted front profile:*

$$\lim_{t\to\infty} |w(t,\cdot) - \phi(\cdot + z_0)|_\lambda = 0. \tag{2.13}$$

In order to prove the above theorem, instead of looking for an appropriate Lyapunov functional (as it was done in [14, 45]) for functional differential equation (2.1), we will use the Berestycki and Nirenberg method of the sliding solutions as well as some ideas of the approach developed by Ogiwara and Matano in [42].

Proof. By Corollary 6, Lemma 3 and Remark 4, solution $w = w(t,z)$ of the initial value problem (2.1), (2.2) has a compact invariant $\omega-$limit set $\omega(w_0)$ such that for some fixed ζ_1, it holds

$$\phi(z - \zeta_1) \leq w_*(0,z) \leq \phi(z + \zeta_1), \quad z \in \mathbb{R}, \quad \text{for each } w_* \in \omega(w_0). \tag{2.14}$$

Then the set

$$A = \{a \in \mathbb{R} : w_*(0,z) \leq \phi(z+a),\ z \in \mathbb{R},\ \text{for each } \ w_* \in \omega(w_0)\}$$

contains ζ_1 and has $-\zeta_1$ as its lower bound. Therefore $\hat{a} = \inf A$ is a well defined finite number. Due to continuity of ϕ, we have that $\hat{a} \in A$ so that

$$w_*(0,z) \le \phi(z+\hat{a}), \ z \in \mathbb{R}, \text{ for each } \ w_* \in \omega(w_0).$$

In fact, since $\omega(w_0)$ is an invariant set, we have that $w_*(t,z) \le \phi(z+\hat{a}), \ z \in \mathbb{R}, \ t \in \mathbb{R}$. Suppose now for a moment that $w_*(0,z') = \phi(z'+\hat{a})$ for some finite z' and some $w_* \in \omega(w_0)$. Therefore, since g is an increasing function, the strong maximum principle yields $w_*(t,z) \equiv \phi(z+\hat{a})$ for all $t \le 0, \ z \in \mathbb{R}$. In particular, $w_*(0,z) \equiv \phi(z+\hat{a})$ so that, for some sequence $t_n \to +\infty$, it holds that $w(t_n+s,z) \to \phi(z+\hat{a})$ uniformly with respect to $s \in [-h,0]$ and z from compact subsets of $\mathbb{R}$. In addition, Corollary 6 allows to evaluate the difference $|w(t_n+s,z) - \phi(z+\hat{a})|/\eta(z)$ in some fixed neighbourhood of the endpoints $z = -\infty$ and $z = +\infty$ and to conclude that $w(t_n+s,z) \to \phi(z+\hat{a}), n \to +\infty$, in the norm $|\cdot|_\lambda$ and uniformly with respect to $s \in [-h,0]$. By Corollary 13, the latter convergence implies (2.13) with $z_0 = \hat{a}$ that completes the proof of the theorem in the case when $w_*(0,z') = \phi(z'+\hat{a})$ holds for some finite z'.

In this way, we are left to consider the situation when

$$w_*(0,z) < \phi(z+\hat{a}), \ z \in \mathbb{R}, \text{ for each } \ w_* \in \omega(w_0). \tag{2.15}$$

In virtue of (2.14), for any given $\delta > 0$, we can find $R > 3ch+1$ sufficiently large to have, for all $w_* \in \omega(w_0)$,

$$w_*(0,z) < \phi(z+\hat{a}) < \delta, \text{ for } z \le -R+ch+1, \quad \phi(z+\hat{a}) > w_*(0,z) > \kappa-\delta, \text{ for } z \ge R-ch-1.$$

Then, using (2.15) and the compactness of the set

$$\{w_*(0,\cdot) : [-R+ch+1, R-ch-1] \to [0,\kappa], \quad w_* \in \omega(w_0)\} \subset C[-R+ch+1, R-ch-1],$$

we deduce the existence of $\varsigma \in (0,1)$ such that

$$w_*(0,z) < \phi(z+\hat{a}-\varsigma),\ z \in [-R+ch+1, R-ch-1],\ w_* \in \omega(w_0).$$

It is clear that

$$\phi(z+\hat{a}) < \kappa < \phi(z+\hat{a}-\varsigma)+\delta, \quad z \geq R-ch.$$

Without the loss of generality, we also can suppose that $\varsigma \in (0,1)$ is such that

$$\phi(z+\hat{a}) < \phi(z+\hat{a}-\varsigma)+\delta e^{\lambda z}, \quad z \leq -R+ch+1.$$

Indeed, observe that $\phi'(z) \leq Ce^{\lambda_2 z},\ z \leq 0$, and therefore, for some $\xi \in (z+\hat{a}-\varsigma, z+\hat{a})$,

$$\phi(z+\hat{a})-\phi(z+\hat{a}-\varsigma) = \phi'(\xi)\varsigma \leq Ce^{\lambda_2(z+\hat{a})}\varsigma \leq \delta e^{\lambda z},\ z \leq 0,$$

once $\varsigma \leq e^{-\lambda_2\hat{a}}\delta/C$. Hence, invoking again the invariance property of $\omega(w_0)$, we can conclude that for each $w_* \in \omega(w_0)$ it holds

$$w_*(t,z) \leq \phi(z+\hat{a}-\varsigma)+\delta\eta(z), \quad t \in \mathbb{R},\ z \in \mathbb{R},$$

and

$$w_*(t,z) \leq \phi(z+\hat{a}-\varsigma),\ z \in [-R+ch+1, R-ch-1],\ t \in \mathbb{R}.$$

By Lemma 2, this yields $w_*(t,z) \leq \phi(z+\hat{a}-\varsigma)+\delta\eta(z)e^{-\gamma t},\ t \geq 0,\ z \in \mathbb{R}$, where $\hat{a}, \varsigma, \gamma$ do not depend on the particular choice of $w_* \in \omega(w_0)$. In consequence, since $w_*(t,z)$ is an entire solution, we obtain that actually $w_*(0,z) \leq \phi(z+\hat{a}-\varsigma),\ z \in \mathbb{R}$, for all $w_* \in \omega(w_0)$. This contradicts to the definition of $\hat{a}$ and shows that the case (2.15) can not happen. □

2.2 Proof of Proposition 2

First, observe that for each g satisfying the assumptions of Proposition 2, we can find a *monotone* function $g_1 : [0,\kappa] \to [0,\kappa]$ possessing all the properties of g and such that $g_1(x) \leq g(x)$. Therefore, in view of the comparison principle, it will not restrict the generality if we will assume additionally the monotonicity of g.

Here, we follow an approach, proposed by Aronson and Weinberger in [2, Theorem 3.1], and based on the maximum principle. In the mentioned work, it was established, for every $\epsilon \in (0, \kappa)$ and appropriate $b_\epsilon > 0$, the existence of a positive solution $q = q(x) \leq \epsilon$ to the Dirichlet boundary value problem

$$q''(x) - q(x) + g(q(x)) = 0, \; x \in I_\epsilon := (0, b_\epsilon),$$
$$q(0) = q(b_\epsilon) = 0.$$

Since we are interested in the asymptotic behavior of $u(t,x) \geq 0$ and $w_0(s,x) \not\equiv 0$, without loss of generality, due to the strong maximum principle we can suppose that $w_0(s,x) > 0$ for all $(s,z) \in \Pi_0$. But then we can choose $\epsilon > 0$ small enough to have $q(x) \leq u(x,s)$ for all $x \in I_\epsilon$, $s \in [-h, 0]$. Let χ_A denote the characteristic function of subset $A \subset \mathbb{R}$. Consider solution $u = u_\epsilon(t,x)$ of the initial value problem $u_\epsilon(s,x) = \chi_{I_\epsilon}(x)q(x)$, $(s,x) \in \Pi_0$, to equation (1.1). Then the difference $\delta(t,x) = q(x) - u_\epsilon(t,x)$ satisfies $\delta(t,0) \leq 0, \; \delta(t, b_\epsilon) \leq 0, \; t \geq 0$, and

$$\delta_t(t,x) - \delta_{xx}(t,x) + \delta(t,x) = g(q(x)) - g(u_\epsilon(t-h,x)) \leq 0, \quad (t,x) \in [0,h] \times I_\epsilon.$$

Hence, by the maximum principle, $u_\epsilon(t,x) \geq q(x)$ on $[0,h] \times I_\epsilon$. Repeating the same argument on $[h, 2h] \times I_\epsilon$, we obtain that $u_\epsilon(t,x) \geq q(x)$ for all $(t,x) \in [h, 2h] \times I_\epsilon$. It is clear that this procedure yields the inequality $q(x) \leq u_\epsilon(t,x) < 1$ in $[0, +\infty) \times I_\epsilon$. But then, since for each positive l, it holds that $\chi_{I_\epsilon}(x)q(x) = u_\epsilon(s,x) \leq u_\epsilon(s+l,x)$, $s \in [-h, 0], \; x \in \mathbb{R}$, we can use the Phragmèn-Lindelöf principle, in order to conclude that $u_\epsilon(t+l,x) \geq u_\epsilon(t,x)$ for all $(t,x) \in [0,h] \times \mathbb{R}$. Similarly to the above analysis, step by step, we can extend the latter inequality for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Thus, for each fixed $x \in \mathbb{R}$, $u_\epsilon(t,x)$ is a non-decreasing bounded function of $t \geq 0$. Let $u_\epsilon(x) = \lim_{t\to+\infty} u_\epsilon(t,x)$, then $u_\epsilon(x) \in (0,\kappa]$ for every $x \in \mathbb{R}$.

Now, a direct application of Lemma 3 shows that $u_\epsilon(x)$ solves

$$u''(x) - u(x) + g(u(x)) = 0, \quad x \in \mathbb{R}$$

while the convergence $u_\epsilon(x) = \lim_{t\to+\infty} u_\epsilon(t,x)$ is uniform on compact subsets of $\mathbb{R}$. Since $g(u) - u > 0$ on $(0,\kappa)$, the function $u_\epsilon(x)$ cannot take (local) minimal values in $(0,\kappa)$. This implies the existence of $u_\epsilon(\pm\infty) \in \{0,\kappa\}$. In other words, $u(x)$ is a positive stationary traveling wave solution of equation (1.1) considered with $h = 0$. It is well known [19] that this is possible only when $u_\epsilon(x) \equiv \kappa$.

Finally, we complete the proof by observing that, due to the maximum principle, it holds $u_\epsilon(t,x) \leq u(t,x)$ on $[0,\infty) \times \mathbb{R}$.

2.3 Proof of Theorem I.3: auxiliary results

In Sections 3.2 and 3.4, we are always assuming that all the conditions of Theorem I.3 are satisfied (recall also that, by simplifying the notation, we write c instead of c_*). The proof of this theorem will follow from a series of lemmas. In the first of them we improve the asymptotic relation $u(t,0) = \kappa + o(1)$ at $+\infty$ known from Proposition 2. As we show below, this convergence is actually of the exponential type.

Lemma 4. *Assume that $\lambda_1 < -\lambda_3$ where λ_3 stands for a unique negative zero of the characteristic function $\chi_\kappa(z,c) := z^2 - cz - 1 + g'(\kappa)e^{-zch}$. If $u(t,x)$ solves (1.1), (1.2) with $w_0 \not\equiv 0$, then there exist numbers $q, \nu > 0$ such that*

$$u(t,0) \geq \kappa - qe^{-\nu t} \quad \textit{for all} \ \ t \geq 0. \tag{2.16}$$

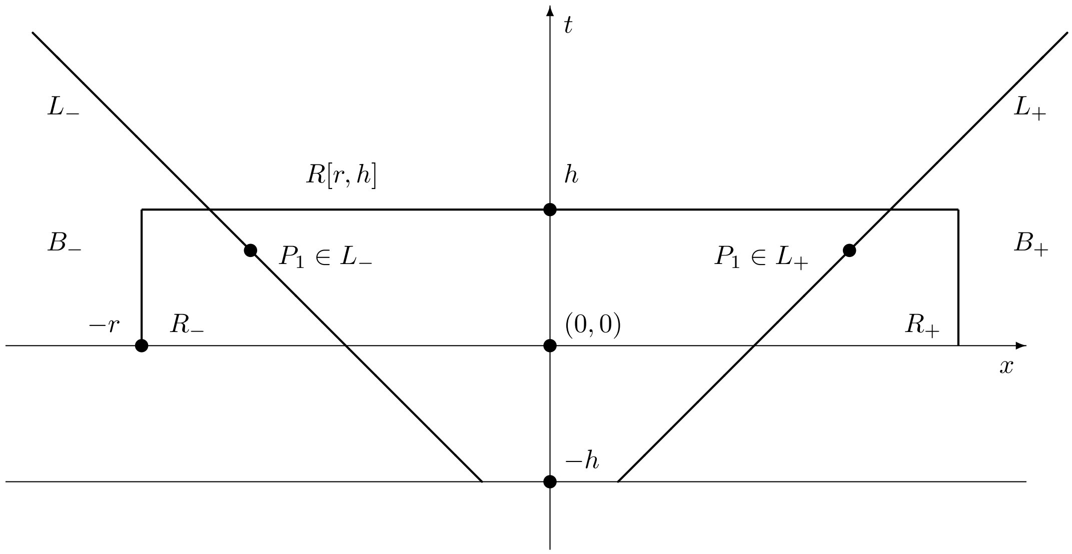

Figure 2.1: Domains $R[r,h], R_1, R_\pm \subset B_\pm$ and lines $L_\pm$.

Proof. First, we fix a positive $\lambda \in (\lambda_1, -\lambda_3) \cap (\lambda_1, \lambda_2)$ and $\gamma < \min\{c\lambda, \gamma_1^*\}$ such that

$$-\lambda^2 + c\lambda + 1 - \gamma - g'(\bar{s})e^{-\lambda ch + \gamma h} > 0 \text{ for } \bar{s} < \delta,$$

where $\delta < \delta_1^*, \gamma_1^*, z_1 < 0 < z_2$ are defined in Steps I, II of Lemma 1. Following [14], we will construct a sub-solution to (1.1) of the form

$$u_-(t,x) = \phi_+(t,x) + \phi_-(t,x) - \kappa - q(t,x),$$

where $\phi_\pm(t,x) = \phi(\pm x + ct - \epsilon(t))$, $q(t,x) = \gamma e^{-\gamma t}\theta(t,x)$ with $\theta(t,x) \le 1$ are defined by

$$\theta(t,x) := \eta(-|x| + ct - \epsilon(\infty) - z_1) = \begin{cases} e^{\lambda(-x+ct-\epsilon(\infty)-z_1)}, \text{ if } (t,x) \in B_+, \\ e^{\lambda(x+ct-\epsilon(\infty)-z_1)}, \text{ if } (t,x) \in B_-, \\ 1, \text{ if } (t,x) \in [-h,\infty) \times \mathbb{R} \setminus (B_+ \cup B_-), \end{cases}$$

$$B_\pm := [-h,\infty) \times \mathbb{R} \cap \{(t,x) : \mp x + ct - \epsilon(\infty) < z_1\},$$

$$L_\pm := [-h,\infty) \times \mathbb{R} \cap \{(t,x) : \mp x + ct - \epsilon(\infty) = z_1\},$$

with an appropriate $\epsilon(t)$ satisfying $\epsilon'(t) > 0$, $\epsilon(t) < 0$. Then $\epsilon(\infty) + z_1 < -ch$ and therefore $B_+ \cap B_- = \emptyset$. See also Figure 1. Set

$$\mathcal{N}_1 u_-(t,x) := (u_-)_t(t,x) - (u_-)_{xx}(t,x) + u_-(t,x) - g(u_-(t-h,x)),$$

$$\tilde{\phi}_\pm(t,x) := \phi(\pm x + c(t-h) - \epsilon(t)) < \phi_\pm(t-h,x).$$

Since $u_-(t,x) = u_-(t,-x)$, it holds that $\mathcal{N}_1 u_-(t,x) = \mathcal{N}_1 u_-(t,-x)$. In view of monotonicity of g and ϕ, we have

$$\begin{aligned}\mathcal{N}_1 u_-(t,x) &\le g(\tilde{\phi}_+(t,x)) + g(\tilde{\phi}_-(t,x)) - g\left(\tilde{\phi}_+(t,x) + \tilde{\phi}_-(t,x) - \kappa - q(t-h,x)\right) \\ &\quad -\epsilon'(t)[\phi'(x+ct-\epsilon(t)) + \phi'(-x+ct-\epsilon(t))] - \kappa - q(t,x) + q_{xx}(t,x) - q_t(t,x).\end{aligned}$$

Claim I: $\mathcal{N}_1 u_-(t,x) = \mathcal{N}_1 u_-(t,-x) < 0$ for $x \ge 0,\ t > 0,\ (t,x) \notin L_\pm$.

By Step II of Lemma 1, $-x + c(t-h) - \epsilon(t) \ge z_2$ implies $\kappa - \delta/2 < \tilde{\phi}_-(t,x)$. Since $x \ge 0$, we also have $\kappa - \delta/2 < \tilde{\phi}_-(t,x) \le \tilde{\phi}_+(t,x)$. By Step I of Lemma 1, for $\gamma \in (0, \gamma_1^*),\ \gamma < \sigma - \delta/2$,

$$\begin{aligned}&g(\tilde{\phi}_-(t,x)) - g(\tilde{\phi}_-(t,x) - [\kappa - \tilde{\phi}_+(t,x) + q(t-h,x)]) \\ &\qquad \le e^{-\gamma h}(1-2\gamma)[\kappa - \tilde{\phi}_+(x,t) + q(t-h,x)].\end{aligned}$$

Hence, since $\theta(t,x)$ is non-decreasing in t, we have, for $t > 0$, that $\mathcal{N}_1 u_-(t,x) \le$

$$\begin{aligned}&(1-2\gamma)e^{-\gamma h}[(\kappa - \tilde{\phi}_+(t,x)) + q(t-h,x)] + g(\tilde{\phi}_+(t,x)) - \kappa - q(t,x) + q_{xx}(t,x) - q_t(t,x) \\ &\quad \le -\tilde{\phi}_+(t,x) + (1-2\gamma)\gamma e^{-\gamma t}\theta(t-h,x) + g(\tilde{\phi}_+(t,x)) - q(t,x) + q_{xx}(t,x) - q_t(t,x) \\ &\quad \le g(\tilde{\phi}_+(t,x)) - \tilde{\phi}_+(t,x) + q(t,x)\begin{cases}\lambda^2 - c\lambda - 1 + \gamma + (1-2\gamma)e^{-\lambda ch}, \text{ if } (t,x) \in B_+, \\ -2\gamma, \text{ if } (t,x) \in [0,\infty) \times \mathbb{R}^+ \setminus B_+.\end{cases}\end{aligned}$$

On the other hand, it is known (see e.g. [55, Remark 1]) that, for some $C > 0$, it holds

$$0 \le \kappa - \tilde{\phi}_+(t,x) \le C e^{-\lambda_3 \epsilon(t)} e^{\lambda_3(x+ct)}, \qquad t \ge -h,\ x \in \mathbb{R}. \tag{2.17}$$

This implies that, for $t > 0, x \geq 0,\ -x + c(t-h) - \epsilon(t) \geq z_2$, it holds that

$$\mathcal{N}_1 u_-(t,x) \leq C e^{-\lambda_3 \epsilon(\infty)} e^{\lambda_3 x} e^{\lambda_3 ct}$$

$$+\gamma e^{-\gamma t} \begin{cases} e^{\lambda(-x+ct-\epsilon(\infty)-z_1)}[\lambda^2 - c\lambda - 1 + \gamma + (1-2\gamma)e^{-\lambda ch}], \text{ if } (t,x) \in B_+, \\ -2\gamma, \text{ if } (t,x) \in [0,\infty) \times \mathbb{R}^+ \setminus B_+, \end{cases}$$

$$\leq e^{-\gamma t} \begin{cases} e^{-\lambda x}[\gamma(\lambda^2 - c\lambda - 1 + \gamma + (1-2\gamma)e^{-\lambda ch}) + C e^{-\lambda_3 \epsilon(\infty)}], \text{ if } (t,x) \in B_+, \\ -2\gamma^2 + C e^{-\lambda_3 \epsilon(\infty)}, \text{ if } (t,x) \in [0,\infty) \times \mathbb{R}^+ \setminus B_+. \end{cases}$$

As a consequence, there exists large negative $\epsilon(\infty)$ (depending on γ and λ_3) such that

$$\mathcal{N}_1 u_-(t,x) < 0 \text{ for } t > 0,\ -x + c(t-h) - \epsilon(t) \geq z_2,\ (t,x) \notin L_+.$$

Next, if $-x + c(t-h) - \epsilon(t) \leq z_1$ then $0 \leq \tilde{\phi}_-(t,x) \leq \delta/2$ and $(t,x) \in B_+$. Thus $\theta(t-h,x) = e^{\lambda(-x+ct-ch-\epsilon(\infty)-z_1)}$ and, for some $\bar{s} < \delta/2$,

$$g(\tilde{\phi}_-(t,x)) - g(\tilde{\phi}_-(t,x) - [\kappa - \tilde{\phi}_+(t,x) + q(t-h,x)]) = g'(\bar{s})[\kappa - \tilde{\phi}_+(t,x) + q(t-h,x)].$$

Thus, recalling that $z_1 < 0$, for large $\epsilon(\infty) < 0$ (which depends on γ and λ_3), we get

$$\mathcal{N}_1 u_-(t,x) \leq g'(\bar{s})[\kappa - \tilde{\phi}_+(t,x) + q(t-h,x)] - q(t,x) + q_{xx}(t,x) - q_t(t,x)$$

$$\leq \gamma e^{-\gamma t} e^{\lambda(-x+ct-\epsilon(\infty)-z_1)}[\lambda^2 - c\lambda - 1 + \gamma + g'(\bar{s})e^{(\gamma - c\lambda)h}] + g'(\bar{s})[\kappa - \tilde{\phi}_+(t,x)]$$

$$\leq \gamma e^{-\gamma t} e^{-\lambda x}[\lambda^2 - c\lambda - 1 + \gamma + g'(\bar{s})e^{(\gamma - c\lambda)h}] + g'(\bar{s}) C e^{\lambda_3(x+ct-\epsilon(t))}$$

$$\leq e^{-\lambda x} e^{-\gamma t} \left\{\gamma[\lambda^2 - c\lambda - 1 + \gamma + g'(\bar{s})e^{(\gamma - c\lambda)h}] + g'(\bar{s}) C e^{-\lambda_3 \epsilon(\infty)}\right\} < 0.$$

Finally, consider $z_1 \leq -x + c(t-h) - \epsilon(t) \leq z_2$. Recall that $\beta > 0$ defined in Step II of Lemma 1 depends only on δ, ϕ and satisfies $\beta < \min_{\zeta \in [z_1, z_2+ch]} \phi'(\zeta)$. Therefore, if we take $\epsilon'(t) = \alpha\gamma e^{-\gamma t}$ for some $\alpha > 0$, then

$$|g(\tilde{\phi}_-(t,x)) - g(\tilde{\phi}_+(t,x) + \tilde{\phi}_-(t,x) - \kappa - q(t-h,x))| \leq L_g[C e^{-\lambda_3 \epsilon(t)} e^{\lambda_3(x+ct)} + q(t-h,x)].$$

In consequence, if α is sufficiently large then $\mathcal{N}_1 u_-(x,t) \leq CL_g e^{\lambda_3(-\epsilon(t)+x+ct)}$

$$+\begin{cases} \gamma e^{-\gamma t}\{-\alpha\beta + e^{\lambda(-x+ct-\epsilon(\infty)-z_1)}[\lambda^2 - c\lambda - 1 + \gamma + L_g e^{(\gamma-\lambda c)h}]\}, \text{ if } (t,x) \in B_+, \\ \gamma e^{-\gamma t}[-\alpha\beta + \gamma - 1 + L_g e^{\gamma h}], \text{ if } (t,x) \in [0,\infty) \times \mathbb{R}^+ \setminus B_+, \end{cases} \leq$$

$$e^{-\gamma t}\left\{\gamma[-\alpha\beta + L_g e^{\gamma h}] + CL_g e^{-\lambda_3\epsilon(\infty)} e^{\lambda_3 x}\right\} < 0, \text{ for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Claim II: There exists $t_0 > 0$ such that $u_-(s,x) \leq u(s+t_0,x)$ for $x \in \mathbb{R},\ s \in [-h,0]$.

Since $\lambda_2 > \lambda$, there exists $r_0 > 0$ depending on $\epsilon(-h), \epsilon(\infty), z_1$ such that, for $s \in [-h,0]$,

$$u_-(s,x) \leq \phi(-|x| + cs - \epsilon(s)) - \gamma\eta(-|x| + cs - z_1 - \epsilon(\infty)) < 0 \text{ if } |x| \geq r_0.$$

Clearly, $u_-(s,x) < \kappa$ for all $|x| \leq r_0, s \in [-h,0]$ and therefore, by Proposition 2, $u_-(s,x) < u(t_0+s,x)$, $|x| \leq r_0, s \in [-h,0]$, for an appropriate $t_0 > 0$.

Claims I and II allow to complete the proof of Lemma 4. First, for $r > 0$, consider rectangle $R[r,h] = [0,h] \times [-r,r]$. Set $\delta(t,x) := u_-(t,x) - u(t+t_0,x)$, the function $\delta(t,x)$ is smooth in $[-h,+\infty) \times \mathbb{R} \setminus \{L_- \cup L_+\}$ (in particular, in the regions $R_\pm = R[r,h] \cap B_\pm$, $R_1 = R[r,h] \setminus (\bar{R}_+ \cup \bar{R}_-)$). Since $\delta(s,x) \leq 0$ in $[-h,0] \times \mathbb{R}$ and

$$\delta_t(t,x) - \delta_{xx}(t,x) + \delta(t,x) \leq g(u_-(t-h,x)) - g(u(t+t_0-h,x)) \leq 0,$$

for all $(t,x) \in [0,h] \times \mathbb{R} \setminus \{L_- \cup L_+\}$, the maximum principle assures that the function $\delta(t,x)$ in $R[r,h]$ is ether negative or it reaches a non-negative maximum at a point $P_1 = (t_1,x_1)$ belonging to $\partial R_1 \cup \partial R_+ \cup \partial R_- \setminus \{h\} \times (-r,r)$. It is easy to see that $P_1 \notin L_\pm$. Indeed, if $P_1 \in L_\pm$ (see Fig. 1) then $\delta_x(P_1+) - \delta_x(P_1-) = \gamma\lambda e^{-\gamma t_1} > 0$. Thus the non-negative maximum of $\delta(t,x)$ on $R[r,h]$ is attained at a point from the parabolic boundary of $R[r,h]$. In consequence, the usual maximum principle holds for each $R[r,h]$ so that, just as it was done in Step V of the proof of Lemma 12, we can appeal to the Phragmèn-Lindelöf principle in order to conclude that $\delta(t,z) \leq 0$

for all $t \in [0,h],\ z \in \mathbb{R}$. Applying the above argument consecutively on the intervals $[h,2h],\ [2h,3h],\ldots$ we find that $\delta(t,x) \leq 0$ for all $t \geq -h, x \in \mathbb{R}$. Therefore, in view of (2.17),

$$u(t+t_0,0) \geq 2\phi(ct-\epsilon(\infty)) - \kappa - \gamma e^{-\gamma t} \geq \kappa - q' e^{-\gamma t}, \quad t \geq -h,$$

for some sufficiently large $q' > \gamma$. Obviously, this yields (2.16) with appropriate $q > q'$. □

Corollary 7. *The conclusion of Lemma 4 holds without the assumption* $\lambda_1 < -\lambda_3$.

Proof. First, we observe that there exists a monotone function $\hat{g}(x) \leq g(x)$ satisfying the hypothesis **(H)** and such that the equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + \hat{g}(u(t-h,x)) \tag{2.18}$$

has a pushed wavefront $\hat{\phi}(\hat{c}t+x)$ with the associated eigenvalues $\hat{\lambda}_1,\ \hat{\lambda}_3 = \lambda_3$ such that $\hat{\lambda}_1 < -\hat{\lambda}_3$. Indeed, let $g_n(x) \leq g(x)$, be a sequence of monotone functions satisfying **(H)**, coinciding with $g(x)$ on $[1/n,\kappa]$, uniformly on $[0,\kappa]$ converging to $g(x)$ and such that $\lim_{n\to+\infty} g_n'(0) = 1$. Then [27, Lemma 3.5] implies that $c_n := c_*(g_n) \leq c := c_*(g)$ while the proof of Proposition 1 shows that $\liminf_{n\to+\infty} c_n \geq c$. This means that $\lim_{n\to+\infty} c_n = c > c_\# > c_\#^{(n)}$ and $\lim_{n\to+\infty} \lambda_1^{(n)} = 0 < -\lambda_3$ where, similarly to $c_\#, \lambda_1$, the numbers $c_\#^{(n)}, \lambda_1^{(n)}$ are determined from the characteristic equation (1.4) with $g'(0)$ replaced by $g_n'(0)$.

In consequence, if $\hat{u}(t,x)$ denotes the solution of the initial value problem (1.2) for (2.18), with $w_0 \not\equiv 0$, then Lemma 4 implies that $\hat{u}(t,0) \geq \kappa - qe^{-\nu t}$, $t > 0$, for some positive q,ν. Finally, by comparing initial value problems (1.1), (1.2) and (2.18), (1.2) and invoking the Phragmèn-Lindelöf principle, we get that $u(t,0) \geq \hat{u}(t,0) \geq \kappa - qe^{-\nu t}$ for all $t > 0$. □

Corollary 8. *Assume that all the conditions of Theorem I.3 are satisfied. Then there exist* $K > 1, t_1 > 0$ *and* $z', z'' \in \mathbb{R}$ *such that*

$$u(t,x) \geq \phi(-|x| + ct - z') - Ke^{-\gamma t}\eta(-|x| + ct - z'') \text{ for all } t > t_1 - h,\ x \in \mathbb{R}.$$

Proof. Consider $u_-(t,x) = \phi(-|x| + ct - \epsilon(t)) - \gamma e^{-\gamma t}\theta(t,x)$. Analyzing the proof of Claim I of Lemma 4, we can easily find that it is also valid for $x \neq 0$ if we replace $\phi_+(t,x)$ with κ. Moreover, in such a case, the restriction $\lambda_1 < -\lambda_3$ is unnecessary (recall that this restriction appears due to the term $\phi_+(t,x) - \kappa$). Hence, we conclude that, for an appropriate choice of $\epsilon(t)$, it holds $\mathcal{N}_1 u_-(t,x) \leq 0$ for all $x > 0,\ t > 0,\ (t,x) \notin L_+$. Since $\mathcal{N}_1 u_-(t,x) = \mathcal{N}_1 u_-(t,-x)$, we conclude that u_- is a sub-solution in the region $x \neq 0,\ t > 0,\ (t,x) \notin L_\pm$. In addition, for some sufficiently large $t_1 > 0$, it holds

$$u(t + t_1, 0) \geq \kappa - qe^{-\gamma(t+t_1)} > \phi(ct - \epsilon(-h)) - \gamma e^{-\gamma t}\eta(ct - \epsilon(\infty) - z_1) =$$

$$\phi(ct - \epsilon(-h)) - \gamma e^{-\gamma t} > u_-(t,0) \text{ for all } t \geq -h.$$

Now, arguing as in Claim II of the proof of Lemma 4, we can also assume that t_1 is chosen in such a way that $u_-(s,x) \leq u(s + t_1, x)$ for $x \in \mathbb{R},\ s \in [-h, 0]$. But then, using the Phragmèn-Lindelöf principle in the regions $[hj, h(j+1)] \times [0, +\infty)$, $[hj, h(j+1)] \times (-\infty, 0]$, $j = 0, 1, \dots$ according to the procedure established in the last paragraph of the proof of Lemma 4, we conclude that, for all $x \in \mathbb{R},\ t \geq t_1 - h$, it holds that

$$u(t,x) \geq u_-(t - t_1, x) = \phi(-|x| + c(t - t_1) - \epsilon(t - t_1)) - \gamma e^{-\gamma(t-t_1)t}\theta(t - t_1, x) \geq$$

$$\phi(-|x| + c(t - t_1) - \epsilon(\infty)) - \gamma e^{-\gamma(t-t_1)t}\eta(-|x| + c(t - t_1) - \epsilon(\infty) - z_1).$$

This completes the proof of the corollary. □

Lemma 5. *Assume all the conditions of Theorem I.3 and suppose that for some sequence* $t_n \to +\infty$ *and* $s_1, s_2 \in \mathbb{R}$*, it holds*

$$\lim_{n\to\infty} \sup_{x\le 0} |u(t_n+s,x) - \phi(x+c(t_n+s)+s_1)|/\eta(x+ct_n) = 0, \tag{2.19}$$

$$\lim_{n\to\infty} \sup_{x\ge 0} |u(t_n+s,x) - \phi(-x+c(t_n+s)+s_2)|/\eta(-x+ct_n) = 0, \tag{2.20}$$

uniformly on $s \in [-h, 0]$*. Then for every* $\delta > 0$ *there exists* $T(\delta) > 0$ *such that*

$$\sup_{x\le 0} \frac{|u(t,x) - \phi(x+ct+s_1)|}{\eta(x+ct)} < \delta \quad \textit{for all } t \ge T(\delta), \tag{2.21}$$

$$\sup_{x\ge 0} \frac{|u(t,x) - \phi(-x+ct+s_2)|}{\eta(-x+ct)} < \delta \quad \textit{for all } t \ge T(\delta).$$

Proof. It suffices to establish (2.21), since $u(t,-x)$ also solves equation (1.1) and satisfies all the hypotheses of Theorem I.3. Without restricting generality, we can take $s_1 = 0$. We know from Corollary 7 that $u(t,0) \ge \kappa - qe^{-\nu t}$, $t \ge 0$. Fix $\gamma \in (0, \min\{\nu, -c\lambda_3\})$ and consider $\epsilon(t) = \alpha\delta\gamma^{-1}e^{-\gamma t}$ (with α defined in Step II of Lemma 1) and

$$u_n(t,x) = \phi(x+ct+ct_n - \alpha\gamma^{-1}e^{\gamma h}\delta + \epsilon(t)) - \delta e^{-\gamma t}\eta(x+ct+ct_n).$$

Let positive integer $N = N(\delta)$ be such that $\delta e^{\nu t_N} > q$ and

$$\sup_{(s,x)\in[-h,0]\times(-\infty,0]} \frac{|u(t_n+s,x) - \phi(x+c(t_n+s))|}{\eta(x+c(t_n+s))} < \delta \text{ for all } n \ge N(\delta).$$

Then we obtain, for all for $(s,x) \in [-h,0]\times(-\infty,0]$,

$$u_N(s,x) \le \phi(x+c(t_N+s)) - \delta\eta(x+c(t_N+s)) \le u(t_N+s,x).$$

Let us show now that a similar relation holds for all $(t,x) \in [t_N,\infty)\times\{0\}$ once $N(\delta)$ is large. Indeed, we have that $u_N(t,0) \le \kappa - \delta e^{-\gamma t}$ for all $t \ge 0$ so that,

$$u(t+t_N,0) - u_N(t,0) \ge \delta e^{-\gamma t} - qe^{-\nu t_N}e^{-\nu t} > 0, \ t \ge 0.$$

Next, observe that $u_n(t,x) = w_-(t, x + c(t+t_n))$ where w_- is defined in Lemma 1 (by Remark 2, the summand $-\alpha\gamma^{-1}e^{\gamma h}$ within the argument of ϕ doesn't matter). Since $\delta < \sigma$, we find that $(u_n)_t(t,x) - (u_n)_{xx}(t,x) + u_n(t,x) - g(u_n(t-h,x)) = (\mathcal{N}w_-)(t, x + c(t+t_n)) < 0$ for all $(t,x) \in [0,\infty) \times \mathbb{R}$, $x + ct + ct_n \neq 0$. Furthermore, if $x' + ct' + ct_n = 0$ at some point (x', t') then $(u_n)_x(t', x'+0) - (u_n)_x(t', x'-0) = \lambda\delta e^{-\gamma t'} > 0$. Therefore, repeatedly applying the Phragmèn-Lindelöf principle in the regions $[hj, h(j+1)] \times (-\infty, 0]$, $j = 0, 1, \dots$ according to the procedure established in the last paragraph of the proof of Lemma 4, we conclude that, for all $x \le 0,\ t \ge -h$,

$$u(t + t_N, x) \ge u_N(t,x) \ge \phi(x + c(t+t_N) - \alpha\gamma^{-1}e^{\gamma h}\delta) - \delta\eta(x + c(t+t_N)).$$

Hence, taking positive constant $K_1 = K_1(\alpha, \gamma, h)$ as in (2.11), we obtain that

$$u(t,x) \ge \phi(x+ct) - \delta(1+K_1)\eta(x+ct),\ t \ge t_N - h,\ x \le 0. \tag{2.22}$$

On the other hand, by our assumptions, for all for $(s,x) \in [-h, 0] \times (-\infty, 0]$,

$$u(t_N + s, x) \le \phi(x + c(t_N + s)) + \delta\eta(x + c(t_N+s)). \tag{2.23}$$

If, in addition, $N = N(\delta)$ is so large that

$$\phi(c(t_N+s)) + \delta\eta(c(t_N+s)) > \kappa,\ s \in [-h, 0],$$

then (2.23) holds also for all $(s,x) \in \Pi_0$. Therefore for $\delta \in (0, q_0]$, by Lemma 1,

$$u(t+t_N, x) \le \phi(x + c(t_N+t) + C\delta) + \delta e^{-\gamma t}\eta(x + c(t_N+t)),\ t \ge 0,\ x \in \mathbb{R},$$

for positive $C > 0$ defined in Lemma 1. Next, due to (2.11), for all $(t,x) \in \mathbb{R}^2$, we have

$$\phi(x + c(t_N+t) + C\delta) \le \phi(x + c(t_N+t)) + K_1\delta\eta(x + c(t_N+t)).$$

In consequence, we obtain

$$u(t,x) \le \phi(x+ct) + \delta(1+K_1)\eta(x+ct),\ \text{for } t \ge t_N,\ x \le 0.$$

The latter inequality together with (2.22) imply (2.21).

□

2.4 Proof of Theorem I.3: main arguments

Set $z = x + ct$ and $w(t,z) := u(t,x) = u(t, z-ct)$, then $w(t,z)$ satisfies equation (2.1), (2.2) for $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$ and possesses a compact and invariant ω-limit set $\omega(w_0)$ defined in Remark 4. Consider the semi-infinite strip $\Omega = \{(s,z) \in \Pi_0,\ z \le -ch\}$. By Corollary 6 and Remark 3, for some $K > 0,\ \zeta_1 \in \mathbb{R}$, it holds

$$w(t,z) \le \phi(z+\zeta_1) + Ke^{-\gamma t}\eta(z+\zeta_1), \quad z \in \mathbb{R},\ t \ge -h. \tag{2.24}$$

Therefore the set

$$A = \{a \in \mathbb{R} : v(s,z) \le \phi(z+a),\ (s,z) \in \Omega,\ \text{for each}\ \ v \in \omega(w_0)\}$$

in non-empty. Since, by Corollary 8,

$$\phi(z-z_1) - K\gamma e^{-\gamma t}\eta(z-z_1) \le w(t,z), \quad z \le ct,\ t \ge -h, \tag{2.25}$$

A is bounded below. Set $\hat{a} := \inf A$, obviously, $\hat{a} \in A$. We claim that $v_*(s_*, z_*) = \phi(z_* + \hat{a})$ for some $(s_*, z_*) \in \Omega$ and $v_* \in \omega(w_0)$. Indeed, suppose on the contrary that

$$v(s,z) < \phi(z+\hat{a}) \text{ for all } (s,z) \in \Omega,\ v \in \omega(w_0). \tag{2.26}$$

For positive ς and an entire solution $v \in \omega(w_0),\ v : \mathbb{R}^2 \to [0,\kappa]$, consider $\rho(t,z) = v(t,z) - \phi(z+\hat{a}-\varsigma)$. Let $R > ch$ be such that $\phi(-R+\zeta_1) < \delta$. Then, for each $\xi(t,z)$ lying between points $v(t-h, z-ch)$ and $\phi(z+\hat{a}-\varsigma-ch)$ with $z \le -R,\ t \in \mathbb{R}$, we have $\xi(t,z) \in (0,\delta)$. Next, set $r(t,z) = \eta(z)e^{-\gamma t}$ and let δ be as in (2.9). In view of (2.9),

$$r_t(t,z) - r_{zz}(t,z) + cr_z(t,z) + r(t,z) = \eta(z)e^{-\gamma t}[1 - \gamma - \lambda^2 + c\lambda]$$

$$\ge \eta(z)e^{-\gamma t}g'(\xi(t,z))e^{-\lambda ch + \gamma h}, \quad t > 0,\ z \le -R,\ \varsigma > 0,\ v \in \omega(w_0),$$

$$\rho_t(t,z) = \rho_{zz}(t,z) - c\rho_z(t,z) - \rho(t,z) + g'(\xi(t,z))\rho(t-h,z-ch), \quad t \in \mathbb{R},\ z \leq -R.$$

On the other hand, since the set $\omega(w_0)$ is compact and invariant (the latter means that $\omega(w_0)$ consists of *entire* solutions $v : \mathbb{R}^2 \to [0,\kappa]$) and ϕ increases on $\mathbb{R}$, we can fix $\varsigma > 0$ such that (3.20) implies

$$(2.27)\ v(t,z) < \phi(z + \hat{a} - \varsigma),\ t > 0,\ -R \leq z \leq -ch,\ v \in \omega(w_0).$$

Without loss of generality, we can also suppose that ς is sufficiently small to meet

$$(2.28) \quad \phi(z+\hat{a}) < \phi(z+\hat{a}-\varsigma) + \eta(z)e^{-\gamma s} \quad \text{for all } z \in \mathbb{R},\ s \in [-h,0].$$

Now, we set $\delta(t,z) := \rho(t,z) - r(t,z)$. Note that, by (3.20) and (2.28), for all for $s \in [-h,0]$, $z \leq -ch$, it holds

$$\delta(s,z) = v(s,z) - (\phi(z+\hat{a}-\varsigma) + \eta(z)e^{-\gamma s}) < v(s,z) - \phi(z+\hat{a}) < 0,$$

and therefore, in virtue of the above mentioned properties of ρ, r,

$$\begin{aligned} &\delta_{zz}(t,z) - \delta_t(t,z) - c\delta_z(t,z) - \delta(t,z) \\ \geq &-g'(\xi(t,z))\rho(t-h,z-ch) + \eta(z)e^{-\gamma t}g'(\xi(t,z))e^{-\lambda ch + \gamma h} \\ &= -g'(\xi(t,z))\delta(t-h,z-ch) > 0 \text{ for } z \leq -R,\ t \in [0,h]. \end{aligned}$$

Taking into account that, due to (2.27), it holds $-\kappa - 1 < \delta(t,z) < 0$ for all $t \in [0,h]$, $-R \leq z \leq -ch$, we can invoke now the Phragmèn-Lindelöf principle [43] in order to conclude that $\delta(t,z) < 0$ for all $t \in [0,h]$, $z \leq -R$. But then, by repeating the above argument for the time intervals $[h,2h]$, $[2h,3h], \dots,$ and using (2.27) we conclude that

$$v(t,z) \leq \phi(z+\hat{a}-\varsigma) + \eta(z)e^{-\gamma t}$$

for all $t \geq 0$, $z \leq -ch$. Due to the invariance property of $\omega(w_0)$ this yields

$$v(s,z) < \phi(z+\hat{a}-\varsigma),\ -h \leq s \leq 0,\ z \leq -ch,\ v \in \omega(w_0),$$

contradicting the definition of $\hat{a}$.

Hence, $w_*(s_*, z_*) = \phi(z_* + \hat{a})$ for some $(s_*, z_*) \in \Omega$ and $w_* \in \omega(w_0)$. Therefore, by the strong principle maximum and invariance property of $\omega(w_0)$, we obtain that $\phi \in \omega(w_0)$.

Next, it follows from (2.24) and (2.25) that, for all $z \leq ct,\ t \geq -h$, it holds

$$|w(t, z) - \phi(z + \hat{a})| \leq \phi(z + \zeta_1) - \phi(z - z_1) + Ke^{-\gamma t}(\eta(z - z_1) + \eta(z + \zeta_1)).$$

In consequence, for each $\epsilon > 0$ we can find $T(\epsilon) > 0$ such that

$$|w(t + s, z) - \phi(z + \hat{a})| < \epsilon \quad \text{for } t \geq T(\epsilon),\ cT(\epsilon) \leq z \leq ct,\ s \in [-h, 0].$$

and

$$\frac{|w(t, z) - \phi(z + \hat{a})|}{\eta(z)} < \epsilon \quad \text{for } t \geq T(\epsilon),\ z \leq -cT(\epsilon),\ s \in [-h, 0].$$

On the other hand, since $\phi \in \omega(w_0)$, there exist $t_n \to \infty$ and an integer $n(\epsilon)$ so that:

$$\frac{|w(t_n + s, z) - \phi(z + \hat{a})|}{\eta(-cM)} < \epsilon, \quad n \geq n(\epsilon),\ |z| \leq cT(\epsilon),\ s \in [-h, 0].$$

Obviously, the last three inequalities imply (2.19). Moreover, by considering the solution $\hat{u}(t, x) = u(t, -x)$ together with the obtained sequence $\{t_n\}$, we can see that (2.20) is also satisfied for a subsequence $\{t_{n_j}\} \subset \{t_n\}$ and an appropriate s_2. Finally, an application of Lemma 5 completes the proof of Theorem I.3.

CHAPTER III

Speed selection problem

3.1 Super- and sub-solutions: definition and properties

The stability analysis of a wavefront $u = \phi(x + ct)$ is usually realised in the co-moving coordinate frame $z = x + ct$ so that $w(t, z) := u(t, z - ct) = u(t, x)$. Clearly, w satisfies the equation

$$w_t(t, z) = w_{zz}(t, z) - cw_z(t, z) - w(t, z) + g(w(t - h, z - ch)), \tag{3.1}$$

while the front profile $\phi(z)$ is a solution of the stationary equation

$$0 = \phi''(z) - c\phi' z - \phi(z) + g(\phi(z - ch)). \tag{3.2}$$

In order to study the front solutions of (3.1), (3.2), different versions of the method of super- and sub- solutions were successfully applied in [31, 49, 55, 63] (in the case of stationary equations similar to (3.2)) and in [7, 32, 49, 51, 58] (in the case of non-stationary equations similar to (3.1)). An efficacious construction of these solutions is the key to the success of this approach. In particular, the studies of front's stability in [32, 58] had used C^3-smooth super- and sub-solutions previously introduced by Chen and Guo in [7, Lemma 3.7]. It is well known that, by cautiously weakening smoothness restrictions, we can improve the overall quality of super- and sub- solutions, cf. [14, 31, 45, 49, 55, 58, 63]. In this paper, inspired by the latter references, we propose to work with somewhat more handy C^1-smooth super- and sub-solutions:

Definition 1. Continuous function $w_+ : [-h,+\infty) \times \mathbb{R} \to \mathbb{R}$ is called a super-solution for (3.1), if, for some $z_* \in \mathbb{R}$, this function is $C^{1,2}$-smooth in the domains $[-h,+\infty) \times (-\infty, z_*]$ and $[-h,+\infty) \times [z_*, +\infty)$ and, for every $t > 0$,

$$\mathcal{N}w_+(t,z) \geq 0, \; z \neq z_*, \text{ while } (w_+)_z(t, z_*-) > (w_+)_z(t, z_*+), \tag{3.3}$$

where the nonlinear operator $\mathcal{N}$ is defined by

$$\mathcal{N}w(t,z) := w_t(t,z) - w_{zz}(t,z) + cw_z(t,z) + w(t,z) - g(w(t-h, z-ch)).$$

The definition of a sub-solution w_- is similar, with the inequalities reversed in (3.3).

The following comparison result is a rather standard one. However, since sub- and super-solutions considered in this paper have discontinuous spatial derivates and, in addition, equation (3.1) contains shifted arguments, we give its proof for the completeness of our exposition. See also [14, 45, 49, 58].

Lemma 1. *Assume* **(H)**. *Let w_+, w_- be a pair of super- and sub-solutions for equation (3.1) such that $|w_\pm(t,z)| \leq Ce^{D|z|}$, $t \geq -h, \; z \in \mathbb{R}$, for some $C, D > 0$ as well as*

$$w_-(s,z) \leq w_0(s,z) \leq w_+(s,z), \quad \textit{for all } s \in [-h, 0], \; z \in \mathbb{R}.$$

Then the solution $w(s,z)$ of equation (3.1) with the initial datum w_0 satisfies

$$w_-(t,z) \leq w(t,z) \leq w_+(t,z) \quad \textit{for all } t \geq -h, \; z \in \mathbb{R}.$$

Proof. In view of the assumed conditions, we have that

$$\pm(g(w_\pm(t-h, z-ch)) - g(w(t-h, z-ch))) \geq 0, \quad t \in [0,h], \; z \in \mathbb{R}.$$

Therefore, for all $t \in (0, h]$, the function $\delta(t, z) := \pm(w(t, z) - w_\pm(t, z))$ satisfies the inequalities

$$\delta(0, z) \leq 0,\ |\delta(t, z)| \leq 2Ce^{D|z|}, \quad \delta_{zz}(t, z) - \delta_t(t, z) - c\delta_z(t, z) - \delta(t, z) =$$
$$\pm(\mathcal{N}w_\pm(t, z) - \mathcal{N}w(t, z) + g(w_\pm(t - h, z - ch)) - g(w(t - h, z - ch))) =$$
$$\pm\mathcal{N}w_\pm(t, z) \pm (g(w_\pm(t - h, z - ch)) - g(w(t - h, z - ch))) \geq 0,\ z \in \mathbb{R} \setminus \{z_*\};$$

$$\frac{\partial\delta(t, z_*+)}{\partial z} - \frac{\partial\delta(t, z_*-)}{\partial z} = \pm\left(\frac{\partial w_\pm(t, z_*-)}{\partial z} - \frac{\partial w_\pm(t, z_*+)}{\partial z}\right) > 0. \tag{3.4}$$

We claim that $\delta(t, z) \leq 0$ for all $t \in [0, h],\ z \in \mathbb{R}$. Indeed, otherwise there exists $r_0 > 0$ such that $\delta(t, z)$ restricted to any rectangle $\Pi_r = [-r, r] \times [0, h]$ with $r > r_0$, reaches its maximal positive value $M_r > 0$ at at some point $(t', z') \in \Pi_r$.

We claim that (t', z') belongs to the parabolic boundary $\partial\Pi_r$ of Π_r. Indeed, suppose on the contrary, that $\delta(t, z)$ reaches its maximal positive value at some point (t', z') of $\Pi_r \setminus \partial\Pi_r$. Then clearly $z' \neq z_*$ because of (3.4). Suppose, for instance that $z' > z_*$. Then $\delta(t, z)$ considered on the subrectangle $\Pi = [z_*, r] \times [0, h]$ reaches its maximal positive value M_r at the point $(t', z') \in \Pi \setminus \partial\Pi$. Then the classical results [43, Chapter 3, Theorems 5,7] show that $\delta(t, z) \equiv M_r > 0$ in Π, a contradiction.

Hence, the usual maximum principle holds for each $\Pi_r,\ r \geq r_0$, so that we can appeal to the proof of the Phragmèn-Lindelöf principle from [43] (see Theorem 10 in Chapter 3 of this book), in order to conclude that $\delta(t, z) \leq 0$ for all $t \in [0, h],\ z \in \mathbb{R}$.

But then we can again repeat the above argument on the intervals $[h, 2h], [2h, 3h], \dots$ establishing that the inequality $w_-(t, z) \leq w(t, z) \leq w_+(t, z),\ z \in \mathbb{R}$, holds for all $t \geq -h$. $\square$

To the best of our knowledge, the following important property of super- (sub-) solutions was first used by Aronson and Weinberger in [2]. See also [49, Proposition 2.9].

Corollary 9. *Assume* **(H)** *and let $w_+(z)$ be an exponentially bounded super-solution for equation (3.1). Consider the solution $w^+(t,z), t \geq 0,$ of the initial value problem $w^+(s,z) = w_+(z)$ for (3.1). Then $w^+(t_1,z) \geq w^+(t_2,z)$ for each $t_1 \leq t_2,\ z \in \mathbb{R}$. A similar result is valid in the case of exponentially bounded sub-solutions $w_-(z)$ which do not depend on t: if $w^-(t,z)$ solves the initial value problem $w^-(s,z) = w_-(z)$ for (3.1), then $w^-(t_1,z) \leq w^-(t_2,z)$ for each $t_1 \leq t_2,\ z \in \mathbb{R}$.*

Proof. We prove only the first statement of the corollary (for super-solution w_+), the case of sub-solution $w_-(z)$ being completely analogous.

By Lemma 1, $w^+(t,z) \leq w_+(z)$ for each $t \geq 0$. Hence, fixing some positive l and considering the initial value problems $u(s,z) = w^+(s+l,z),\ v(s,z) = w_+(z)$, $s \in [-h,0],\ z \in \mathbb{R}$, for equation (3.1), we find that $u(t,z) = w^+(t+l,z) \leq v(t,z) = w^+(t,z)$, $t > 0,\ z \in \mathbb{R}$. □

3.2 Proof of Theorem I.5 and Corollary 1

In this section, we take some $c \geq c_\#$ and assume the conditions of Theorem I.5. This result will follow from Theorem 1 proved below. Everywhere in the section we denote by $w(t,z)$ solution of equation (3.1) satisfying the initial value condition $w(s,z) = w_0(s,z)$, $(s,z) \in \Pi_0$.

It is easy to see that, given $q^* > 0,\ q_* \in (0,\kappa)$, there are $\delta^* < \delta_0$, $\gamma^* > 0$ such that

$$g(u) - g(u - qe^{\gamma h}) \leq q(1-2\gamma),$$
$$(u,q,\gamma) \in \Pi_- = [\kappa - \delta^*, \kappa] \times [0, q_*] \times [0, \gamma^*]; \tag{3.5}$$

$$g(u) - g(u + qe^{\gamma h}) \geq -q(1-2\gamma),$$
$$(u,q,\gamma) \in \Pi_+ = [\kappa - \delta^*, \kappa] \times [0, q^*] \times [0, \gamma^*]. \tag{3.6}$$

Indeed, it suffices to note that the continuous functions

$$G_-(u,q,\gamma) := \begin{cases} 1 + (g(u - e^{\gamma h}q) - & g(u))/q, \ \ (u,q,\gamma) \in \Pi_-; \\ 1 - e^{\gamma h}g'(u), & u \in [\kappa - \delta^*, \kappa], \ q = 0, \ \gamma \in [0, \gamma^*], \end{cases}$$

$$G_+(u,q,\gamma) := \begin{cases} 1 - (g(u + e^{\gamma h}q) - & g(u))/q, \ \ (u,q,\gamma) \in \Pi_+; \\ 1 - e^{\gamma h}g'(u), & u \in [\kappa - \delta^*, \kappa], \ q = 0, \ \gamma \in [0, \gamma^*], \end{cases}$$

are positive on $\Pi_\pm$ provided that γ^*, δ^* are sufficiently small.

From now on, we fix $\gamma \in [0, \gamma_*)$, $\delta \in (0, \delta^*)$ such that (3.5) and (3.6) hold and

$$-\gamma + c\lambda - \lambda^2 + 1 - g'(0)e^{\gamma h}e^{-\lambda ch} \geq 0.$$

It is easy to see that $\gamma = 0$ for $\lambda = \lambda_1(c)$ while γ can be chosen positive if $\lambda \in (\lambda_1(c), \lambda_2(c))$. Consider b determined by the equation $\phi(b - ch) = \kappa - \delta^*/2$. Without loss of generality we can assume that $b > 0$.

Lemma 2. *Suppose that $L_g = g'(0)$ in ($\boldsymbol{H}$). Let $\gamma \geq 0$ be as defined above. If either $c > c_\#$ with $\lambda \in (\lambda_1(c), \lambda_2(c))$ or $c \geq c_\#$ with $\lambda = \lambda_1(c)$, then*

$$w_0(s,z) \leq \phi(z) + q\eta_\lambda(z-b), \quad z \in \mathbb{R}, \quad s \in [-h, 0],$$

with $q \in (0, q^*]$ implies

$$w(t,z) \leq \phi(z) + qe^{-\gamma t}\eta_\lambda(z-b), \quad z \in \mathbb{R}, \quad t \geq -h.$$

Similarly, the inequality

$$\phi(z) - q\eta_\lambda(z-b) \leq w_0(s,z), \quad z \in \mathbb{R}, \quad s \in [-h, 0],$$

with some $0 < q \leq q_*$ implies

$$\phi(z) - qe^{-\gamma t}\eta_\lambda(z-b) \leq w(t,z), \quad z \in \mathbb{R}, \quad t \geq -h.$$

Each conclusion of the lemma holds without any upper restriction on the size of q if we replace $\eta_\lambda(z-b)$ with $\xi(z,\lambda) = \exp(\lambda z)$.

Proof. Set $w_\pm(t,z) = \phi(z) \pm qe^{-\gamma t}\eta_\lambda(z-b)$. Then, for $t > 0$ and $z \in \mathbb{R} \setminus \{b\}$, after a direct calculation we find that

$$\mathcal{N}w_\pm(t,z) = \pm qe^{-\gamma t}[-\gamma\eta_\lambda(z-b) + c\eta'_\lambda(z-b) - \eta''_\lambda(z-b) + \eta_\lambda(z-b)] +$$

$$g(\phi(z-ch)) - g(w_\pm(t-h, z-ch)).$$

It is clear that for $z < b$ (if we are considering $\eta_\lambda(z-b)$) as well as for all $z \in \mathbb{R}$ (if we are using $\xi(z,\lambda)$ instead of $\eta_\lambda(z-b)$), it holds that

$$\pm\mathcal{N}w_\pm(t,z) \geq qe^{-\gamma t}e^{\lambda(z-b)}[-\gamma + c\lambda - \lambda^2 + 1 - g'(0)e^{\gamma h}e^{-\lambda ch}] \geq 0.$$

If $z > b$ and $q \in (0, q^*]$, then (3.6) implies

$$\mathcal{N}w_+(t,z) \;\; \geq \;\; qe^{-\gamma t}[-\gamma + 1 - (1-2\gamma)] = \gamma qe^{-\gamma t} > 0.$$

Similarly, if $z > b$ and $q \in (0, q_*]$, we obtain from (3.5) that

$$-\mathcal{N}w_-(t,z) \;\; \geq \;\; qe^{-\gamma t}[-\gamma + 1 - (1-2\gamma)] = \gamma qe^{-\gamma t} > 0.$$

Next, since

$$\pm\left(\frac{\partial w_\pm(t,b+)}{\partial z} - \frac{\partial w_\pm(t,b-)}{\partial z}\right) = -q\lambda e^{-\gamma t} < 0,$$

we conclude that $w_\pm(t,z)$ is a pair of super- and sub-solutions for equation (3.1). Finally, an application of Lemma 1 completes the proof. □

Lemma 2 implies that front solutions of equation (1.1) are locally stable:

Corollary 10. *Let the triple $(c,\lambda,\gamma) \in [c_\#, +\infty) \times [\lambda_1(c), \lambda_2(c)) \times \mathbb{R}_+$ be as in Lemma 2 and suppose that*

$$\sup_{s\in[-h,0]} |\phi(\cdot) - w_0(s,\cdot)|_\lambda < \rho e^{-\lambda b}$$

for some $\rho < \kappa$. Then

$$|\phi(\cdot) - w(t,\cdot)|_\lambda < \rho e^{-\gamma t}, \;\; t \geq 0.$$

Proof. The statement of the corollary is an immediate consequence of Lemma 2, since, due to our assumptions, for all $z \in \mathbb{R},\ s \in [-h, 0]$,

$$\phi(z) - \rho\eta_\lambda(z-b) \le \phi(z) - \rho e^{-\lambda b}\eta_\lambda(z) \le w_0(s,z) \le$$

$$\phi(z) + \rho e^{-\lambda b}\eta_\lambda(z) \le \phi(z) + \rho\eta_\lambda(z-b).$$

□

We note that assumption $(IC1')$ allows consideration of initial functions w_0 which can be equal to 0 on compact subsets of Π_0. This fact complicates the construction of adequate sub-solutions. In the next assertion we show that, without restricting generality, the positivity of w_0 can assumed in our proofs.

Corollary 11. *Suppose that $L_g = g'(0)$ in (**H**) and that $w_0(s,z)$, $(s,z) \in \Pi_0$, satisfies the assumptions (IC1'), (IC2'). Then the following holds.*

A. *If $c \ge c_\#$ and*

(3.7) $$\lim_{z\to-\infty} w_0(s,z)/\phi(z) = 1,$$

uniformly on $s \in [-h, 0]$, then $w(2h+s, z) > 0,\ (s,z) \in \Pi_0$, also satisfies the assumptions (IC1), (IC2) and $\lim_{z\to-\infty} w(t,z)/\phi(z) = 1$ uniformly with respect to $t \in [0, +\infty)$.

B. *Suppose that $c > c_\#$, $\lambda \in (\lambda_1(c), \lambda_2(c))$ together with*

(3.8) $$q_0 := \sup_{s\in[-h,0]} |\phi(\cdot) - w_0(s,\cdot)|_\lambda < \infty.$$

Then $w(2h+s, z) > 0,\ (s,z) \in \Pi_0$, also satisfies the assumptions (IC1), (IC2) and, for each $t \ge 0$,

$$\sup_{s\in[-h,0]} |\phi(\cdot) - w(s+t,\cdot)|_\lambda < \infty.$$

Proof. The positivity of $w(2h+s,z)$ for $(s,z)\in\Pi_0$, is obvious. Next, the fulfilment of separation condition $(IC2')$ for $w(2h+s,z)$ can be proved similarly to [51, Proposition 1.2] (alternatively, the reader can use Duhamel's formula). Next, since $w\equiv 0$ and $w\equiv\max\{\kappa,|w_0|_\infty\}$ are, respectively, sub- and super-solutions of equation (3.1), the condition $(IC1')$ is also fulfilled. Finally, the proofs of the persistence of properties (3.7) and (3.8) are given below.

A. Set $\lambda_c=\lambda_1$ if $c=c_\#$ or fix some $\lambda_c\in(\lambda_1(c),\lambda_2(c))$ if $c>c_\#$. It follows from (3.7) that for every $s\in\mathbb{R}$, it holds

$$\lim_{z\to-\infty} w_0(s,z)/\phi(z+s)=e^{\lambda_1 s}$$

uniformly on $s\in[-h,0]$. Therefore, for each small $\delta>0$ there exists a large $q=q(\delta,w_0)>0$ such that

$$\phi(z-\delta)-q\xi(z,\lambda_c)\le w_0(s,z)\le\phi(z+\delta)+q\xi(z,\lambda_c),\ \ (s,z)\in\Pi_0. \tag{3.9}$$

Then Lemma 2 assures that

$$\phi(z-\delta)-q\xi(z,\lambda_c)\le w(t,z)\le\phi(z+\delta)+q\xi(z,\lambda_c),\ \ t\ge 0,\ \ z\in\mathbb{R},$$

so that, for all $t\ge 0$ and $z\in\mathbb{R}$, it holds

$$\mathfrak{l}(z,\delta):=\frac{\phi(z-\delta)}{\phi(z)}-1-q\frac{\xi(z,\lambda_c)}{\phi(z)}\le\frac{w(t,z)}{\phi(z)}-1\le\mathfrak{r}(z,\delta):=\frac{\phi(z+\delta)}{\phi(z)}-1+q\frac{\xi(z,\lambda_c)}{\phi(z)}.$$

Now, since

$$\lim_{z\to-\infty}\mathfrak{l}(z,\delta)=e^{-\lambda_1\delta}-1,\qquad\lim_{z\to-\infty}\mathfrak{r}(z,\delta)=e^{\lambda_1\delta}-1,$$

for each $\epsilon>0$ we can indicate $\delta=\delta(\epsilon)$ and z_ϵ such that

$$-\epsilon\le\frac{w(t,z)}{\phi(z)}-1\le\epsilon\quad\text{for all } t\ge 0\ \text{ and } z\le z_\epsilon.$$

B. We have that

$$\phi(z)-q_0\xi(z,\lambda)\le w_0(s,z)\le\phi(z)+q_0\xi(z,\lambda),\ \ (s,z)\in\Pi_0,$$

so that the last conclusion of the corollary follows from Lemma 2. □

Remark 5. Corollary 11A shows that asymptotic relation (3.7) is a time invariant of $w(t, z)$. In the next section, Lemma 4 gives an amplified version of this result.

Theorem 1. *In addition to (**H**), suppose that $L_g = g'(0)$. If the initial function w_0 satisfies the assumptions (IC1'), (IC2'), then the following holds.*

A. Take $c \geq c_{\#}$ and assume (3.7). Then

$$|w(t, \cdot)/\phi(\cdot) - 1|_0 = o(1),\ t \to +\infty.$$

B. If $c > c_{\#}$ and $\lambda \in (\lambda_1(c), \lambda_2(c))$ then (3.8) implies

$$|\phi(\cdot) - w(t, \cdot)|_\lambda \leq Ce^{-\gamma t},\ t \geq 0,$$

for some positive C, γ (in fact, $\gamma > 0$ can be chosen as in Lemma 2).

Proof. In virtue of Corollary 11, without loss of generality, we can assume that $w_0(s, z) > 0$ on Π_0.

A. As in the proof of Corollary 11A, set $\lambda_c = \lambda_1$ if $c = c_{\#}$ or take some $\lambda_c \in (\lambda_1(c), \lambda_2(c))$ if $c > c_{\#}$. We know from Lemma 2 that the functions $\phi(z) \pm q\xi(z, \lambda_c)$ constitute a pair of super- and sub-solutions for equation (3.1) for each positive q. The main drawback of these solutions is their unboundedness. Hence, first we show how to correct this deficiency of $\phi(z) \pm q\xi(z, \lambda_c)$.

So, fix $\delta > 0$ and take $q = q(\delta, w_0) > 0$ large enough to meet (3.9). Let $(-\infty, p)$ be the maximal interval where the function $\phi(z - \delta) - q\xi(z, \lambda_c)$ is positive. Then, for sufficiently small $\epsilon \in (0, \kappa)$, the equation

$$\phi(z - \delta) - q\xi(z, \lambda_c) = \epsilon$$

has exactly two solutions $z_1(\epsilon) < z_2(\epsilon)$ on $(-\infty, p)$. It holds that $z_1(0+) = -\infty$, $z_2(0+) = p$ and therefore we can find $\epsilon > 0$ such that $z_2(\epsilon) - z_1(\epsilon) > ch$ and

$$\inf\{w_0(s,z) : z \geq z_1(\epsilon),\ s \in [-h, 0]\} > \epsilon.$$

It is easy to see that the functions

$$w_-(z) := \begin{cases} \phi(z-\delta) - q\xi(z, \lambda_c), & z \leq z_2(\epsilon), \\ \epsilon, & z_2(\epsilon) \leq z, \end{cases}$$

and

$$w_+(z) := \min\{\kappa + |w_0|_\infty, \phi(z+\delta) + q\xi(z, \lambda_c)\}$$

satisfy

$$w_-(z) \leq w_0(s,z) \leq w_+(z),\ (s,z) \in \Pi_0,$$

and that they are, respectively, a sub-solution and a super-solution for equation (3.1). Thus Corollary 9 implies that

$$w_-(z) \leq w^-(t,z) \leq w(t,z) \leq w^+(t,z) \leq w_+(z), \tag{3.10}$$

where $w^\pm(t,z)$ denote the solutions of (3.1) satisfying the initial conditions $w^\pm(s,z) = w_\pm(z)$, $z \in \mathbb{R}$, $s \in [-h, 0]$. From Corollary 9 we also obtain that $w^\pm(t,z)$ converge (uniformly on compact subsets of $\mathbb{R}$) to some functions $\phi^\pm(z)$ such that

$$w^-(z) \leq \phi^-(z) \leq \phi^+(z) \leq w^+(z).$$

It is well known (see e.g. [51, Lemma 2.8]) that $\phi^\pm$ satisfy the profile equation (3.2). Since $\phi^\pm$ are positive and bounded, $\phi(-\infty) = 0$ and $\liminf_{z\to+\infty} \phi(z) > 0$, we conclude from [55, Proposition 2 and Theorem 1.2] that $\phi^\pm(z) = \phi(z \pm \delta_\pm)$, $z \in \mathbb{R}$ for some $-\delta \leq \delta_- \leq \delta_+ \leq \delta$.

Furthermore, we claim that

$$w^* := \limsup_{t\to+\infty} |w^+(t,\cdot)|_\infty \le \kappa, \quad w_* := \lim_{(T,Z)\to+\infty} \inf_{z\ge Z, t\ge T} w^-(t,z) = \kappa.$$

Clearly, $w_* \le w^*$. To prove that $w^* \le \kappa$, it suffices to observe that the homogeneous solution $w_g(t),\ t \ge 0$, of equation (3.1) defined as the solution of the initial value problem

$$w'(t) = -w(t) + g(w(t-ch)),\ w_g(s) = |w_0|_\infty + \kappa,\ s \in [-h, 0],$$

dominates w^+ (i.e. $w^+(t,z) \le w_g(t)$ for all $z \in \mathbb{R}$, $t \ge -h$) in view of Lemma 1 and converges to κ.

Next, suppose that $w_* < \kappa$ and take Z, T so large and $\delta_1 > \varepsilon_1 > 0$ so small that

(i) $w^-(t,z) > w_* - \delta_1$ for all $z \ge Z - ch,\ \ t \ge T - h$;

(ii) $w^-(t,z) > \kappa - \varepsilon_1$ for all $t \ge T - h$, and $z \in [Z - ch, Z]$;

(iii) homogeneous solution $w_h(t),\ t \ge 0$, of equation (3.1) defined as the solution of the initial value problem

$$w'(t) = -w(t) + g(w(t-ch)),\ w_h(s) = w_* - \delta_1,\ s \in [-h, 0],$$

satisfies the inequalities

$$w_h(t) \le (w_* + \kappa)/2,\ t \in [-h, a_1],$$

$$(w_* + \kappa)/2 \le w_h(t \le w_h(a_2) = \kappa - \varepsilon_1,\ t \in [a_1, a_2],$$

for sufficiently large $a_2 > a_1 + h > h$ (observe here that from [**?**, Corollary 2.2, p. 82] we know that $w_h(t)$ converges monotonically to κ). Therefore, for each $T_1 \ge T$ and all $t \in (T_1, T_1 + h]$, $z \ge Z$, the function $\delta(t,z) = w_h(t - T_1) - w^-(t,z)$ satisfies the inequalities

$$|\delta(t,z)| \le \kappa, \quad \delta_{zz}(t,z) - \delta_t(t,z) - c\delta_z(t,z) - \delta(t,z) =$$

$$g(w^-(t-h, z-ch)) - g(w_h(t - T_1 - h)) > 0.$$

In addition, we have that

$$\delta(T_1, z) < 0, z \geq Z, \quad \delta(t, Z) < 0,\ t \in [T_1, T_1 + a_2].$$

In consequence, by the Phragmèn-Lindelöf principle,

$$\delta(t, z) = w_h(t - T_1) - w^-(t, z) \leq 0, \text{ for all } t \in [T_1, T_1 + h],\ z \geq Z.$$

It is clear that, using step by step integration method, we can repeat the above procedure till the maximal moment t_* before which the inequality $g(w^-(t-h, z-ch)) \geq g(w_h(t - T_1 - h))$ for $z \geq Z$ is preserved. Therefore

$$w_h(t - T_1) \leq w^-(t, z) \text{ for all } t \in [T_1, T_1 + a_2],\ z \geq Z,$$

so that

$$(w_* + \kappa)/2 \leq w^-(t, z) \text{ for all } t \in [T_1 + a_1, T_1 + a_2],\ z \geq Z.$$

However, since $T_1 \geq T$ is an arbitrarily chosen number, we conclude that $w_* \geq (w_* + \kappa)/2$, contradicting to our initial assumption that $w_* < \kappa$. Hence $w^{\pm}(t, z) \to \phi^{\pm}(z)$ as $t \to +\infty$ uniformly on $\mathbb{R}$. In virtue of (3.10), we obtain

$$\limsup_{t \to +\infty} |w(t, \cdot)/\phi(\cdot) - 1|_0 \leq e^{\lambda_1 \delta} - 1,$$

for each small δ. This completes the proof of the first part of Theorem 1.

B. We deduce from (3.8) that

$$\phi(z) - q_0 e^{\lambda b} \xi(z - b, \lambda) \leq w_0(s, z) \leq \phi(z) + q_0 e^{\lambda b} \xi(z - b, \lambda), \quad z \in \mathbb{R},\ s \in [-h, 0].$$

As a consequence, Lemma 2 guarantees that, for some positive γ and all $z \in \mathbb{R}$, $t \geq -h$,

$$\phi(z) - q_0 e^{\lambda b} e^{-\gamma t} \xi(z - b, \lambda) \leq w(t, z) \leq \phi(z) + q_0 e^{\lambda b} e^{-\gamma t} \xi(z - b, \lambda).$$

From the part A of this theorem, we also know that $\lim_{t\to+\infty} w(t,z) = \phi(z)$ uniformly on $\mathbb{R}$. Therefore there exist a large $T_1 > 0$ and positive $q_2 < \min\{q^*, q_*\}$ such that, for all $z \in \mathbb{R}$, $t \geq T_1 - h$,

$$\phi(z) - q_2\eta_\lambda(z-b) \leq w(t,z) \leq \phi(z) + q_2\eta_\lambda(z-b).$$

Again applying Lemma 2, we obtain that

$$\phi(z) - q_2 e^{-\gamma(t-T_1)}\eta_\lambda(z-b) \leq w(t,z) \leq \phi(z) + q_2 e^{-\gamma(t-T_1)}\eta_\lambda(z-b) \quad t > T_1,\ z \in \mathbb{R}.$$

Thus

$$|\phi(z) - w(t,z)|_\lambda \leq (q_2 e^{\gamma T_1})e^{-\gamma t},\ t \geq T_1,$$

that proves the second statement of the theorem. □

3.3 Stability lemma and invariance of the leading asymptotic term

In this section, we are presenting two auxiliary results which are necessary to prove Theorem I.4. First we demonstrate a quite general local stability lemma. Here $\delta^*, b > 0$ are as at the beginning of Section 3.2.

Lemma 3. *Assume that $c > c_*$ and write, for short, $\eta_1(z) = \min\{1, e^{\lambda_1(c)z}\}$ instead of $\eta_{\lambda_1}(z)$. Then*

$$w_\pm(t,z) := \phi(z \pm \epsilon_\pm(t)) \pm q e^{-\gamma t}\eta_1(z),\ q \in (0, \min\{q^*, q_*\}],$$

are super- and sub-solutions for appropriately chosen functions

$$\epsilon_+(t) := \frac{\alpha q}{\gamma}(e^{\gamma h} - e^{-\gamma t}) > 0, \quad \epsilon_-(t) := -\frac{\alpha q}{\gamma}e^{-\gamma t} < 0, \quad t > -h.$$

The parameters α, $\gamma > 0$ are fixed later in the proof and depend only on g, ϕ, c, h, λ_1.

Proof. Set $z_* = 0$ and observe that the smoothness conditions and the second inequality in (3.3) are satisfied in view of

$$\pm\left(\frac{\partial w_\pm(t,0+)}{\partial z} - \frac{\partial w_\pm(t,0-)}{\partial z}\right) = -q\lambda_1(c)e^{-\gamma t} < 0.$$

In order to establish the first inequality of (3.3), we proceed with the following direct calculation:

$$\pm\mathcal{N}w_\pm(t,z) := \epsilon'_\pm(t)\phi'(z\pm\epsilon_\pm(t)) - \gamma q e^{-\gamma t}\eta_1(z) \mp \phi''(z\pm\epsilon_\pm(t)) - qe^{-\gamma t}\eta_1''(z)$$

$$\pm c\phi'(z\pm\epsilon_\pm(t)) + cqe^{-\gamma t}\eta_1'(z) \pm \phi(z\pm\epsilon_\pm(t)) + qe^{-\gamma t}\eta_1(z) \mp g(w_\pm(t-h,z-ch)) \geq$$

$$\alpha q e^{-\gamma t}\phi'(z\pm\epsilon_\pm(t)) - \gamma q e^{-\gamma t}\eta_1(z) + cqe^{-\gamma t}\eta_1'(z) + qe^{-\gamma t}\eta_1(z) - qe^{-\gamma t}\eta_1''(z)$$

$$\pm\left(g(\phi(z-ch\pm\epsilon_\pm(t))) - g(\phi(z-ch\pm\epsilon_\pm(t)) \pm qe^{-\gamma(t-h)}\eta_1(z-ch))\right),\ z\neq 0.$$

Here we are using the fact that $g, \phi, \epsilon_\pm$ are strictly increasing functions.

From now on, we will fix d and α defined by

$$d := \inf_{z\leq b}\phi'(z)/\eta_1(z) > 0 \quad \text{and} \quad \alpha := d^{-1}e^{\gamma h}L_g. \tag{3.11}$$

Note that α, d, γ depend only on g, ϕ, c, h, λ.

We claim that $\pm\mathcal{N}w_\pm(t,z) \geq 0$ for all $z \neq 0,\ t\geq 0$ and $q \in \mathbb{R}_+$.

Indeed, suppose first that $z\pm\epsilon_\pm(t) \leq b$. Then, chosing positive parameter $\gamma < g'(0)e^{-\lambda_1 ch}$, we find that

$$0 \geq \pm\left(g(\phi(z-ch\pm\epsilon_\pm(t))) - g(\phi(z-ch\pm\epsilon_\pm(t))) \pm qe^{-\gamma(t-h)}\eta_1(z-ch)\right) \geq$$

$$-L_g q e^{-\gamma(t-h)}\eta_1(z-ch), \quad \pm\mathcal{N}w_\pm(t,z) \geq$$

$$qe^{-\gamma t}\left\{\eta_1(z\pm\epsilon_\pm(t))d\alpha + ([1-\gamma]\eta_1(z) + c\eta_1'(z) - \eta_1''(z) - e^{\gamma h}L_g\eta_1(z-ch))\right\}$$

$$\geq qe^{-\gamma t}\left(\eta_1(z\pm\epsilon_\pm(t))d\alpha - e^{\gamma h}L_g\eta_1(z-ch)\right) > 0.$$

Similarly, if $z \pm \epsilon_\pm(t) \geq b$, then invoking (3.6) we obtain, for all $t \geq 0$, that

$$0 \geq \pm\left(g(\phi(z - ch \pm \epsilon_\pm(t))) - g(\phi(z - ch \pm \epsilon_\pm(t))) \pm qe^{-\gamma(t-h)}\eta_1(z - ch)\right) \geq$$

$$-qe^{-\gamma t}\eta_1(z - ch)(1 - 2\gamma), \quad \pm\mathcal{N}w_\pm(t, z) \geq$$

$$qe^{-\gamma t}\left([1 - \gamma]\eta_1(z) + c\eta_1'(z) - \eta_1''(z) - (1 - 2\gamma)\eta_1(z - ch)\right) \geq$$

$$qe^{-\gamma t}\left\{\begin{array}{ll} e^{\lambda_1 z}[1 - \gamma + c\lambda_1 - \lambda_1^2 - e^{-\lambda_1 ch}(1 - 2\gamma)], & z < 0 \\ \gamma, & z > 0 \end{array}\right\} > 0.$$

The proof of Lemma 3 is completed. □

Corollary 12. *Let $\gamma > 0$ be as in Lemma 3 and α be as in (3.11). If $c > c_*$ and non-negative initial function w_0 satisfies*

$$\phi(z) - q_-\eta_1(z) \leq w_0(s, z) \leq \phi(z) + q_+\eta_1(z), \quad z \in \mathbb{R}, \quad s \in [-h, 0],$$

for some $0 < q_\pm < \varsigma_0 := \min\{\gamma, \min\{q_*, q^*\}\exp\left(-\lambda_1\alpha e^h\right)\}$, then there exists positive number $C = C(g, \phi)$ such that

$$\phi(z - Cq_-) - Cq_-e^{-\gamma t}\eta_1(z) \leq w(t, z) \leq \phi(z + Cq_+) + Cq_+e^{-\gamma t}\eta_1(z), \; z \in \mathbb{R}, t \geq -h.$$

Proof. The right hand side inequality is a direct consequence of Lemmas 1 and 3 in view of the estimations

$$w_0(s, z) \leq \phi(z) + q_+\eta_1(z) \leq \phi(z + \epsilon_+(s)) + q_+e^{-\gamma s}\eta_1(z), \; (z, s) \in \Pi_0.$$

Since $\epsilon_+(t)$ increases on $\mathbb{R}$, this proves conclusion of the lemma with $C = C_1 := \epsilon(\infty) = \alpha e^{\gamma h}/\gamma$.

In order to prove the left hand side inequality, observe that

$$w_0(s, z - \epsilon_-(-h)) \geq \phi(z - \epsilon_-(s)) - q_-e^{-\lambda_1\epsilon_-(-h)}e^{-\gamma s}\eta_1(z) \geq$$

$$\phi(z - \epsilon_-(s)) - 0.5e^{-h}\delta e^{-\gamma s}\eta_1(z), \quad (z, s) \in \Pi_0.$$

This implies that, for all $t \geq -h$, $z \in \mathbb{R}$, it holds

$$w(t,z) \geq \phi(z - \epsilon_+(t)) - q_- e^{-2\lambda_1 \epsilon_-(-h)} e^{-\gamma t} \eta_1(z) \geq$$

$$\phi(z - C_1 q_-) - C_2 q_- e^{-\gamma t} \eta_1(z), \; C_2 := \exp\left(2\lambda_1 \alpha e^h\right).$$

Setting $C = \max\{C_1, C_2\}$, we complete the proof of Corollary 12. □

Corollary 13. *For every $\epsilon > 0$ there exists $\varsigma(\epsilon) > 0$ such that*

$$|\phi(\cdot) - w(s,\cdot)|_{\lambda_1} < \varsigma(\epsilon), \quad s \in [-h, 0],$$

implies that $|\phi(\cdot) - w(t,\cdot)|_{\lambda_1} < \epsilon$ for all $t \geq 0$.

Proof. It suffices to take

$$\varsigma(\epsilon) = \min\left\{\varsigma_0, \frac{\epsilon}{C(1 + e^{\lambda_1 C \varsigma_0} \sup_{z \in \mathbb{R}} [\phi'(z)/\eta_1(z)])}\right\}$$

and to apply Corollary 12. □

The second main result of this section assures the invariance of the main asymptotic term at $-\infty$ of solutions with 'good' initial data.

Lemma 4. *Suppose that the birth function g is bounded and that there exists $g'(0) > 1$. If the initial fragment $u(s,z)$ of a bounded solution $u(t,z)$ to equation (1.1) is such that, for some positive eigenvalue $\lambda_j(c)$, $j = 1, 2$, it holds that $u(s, x - cs)e^{-\lambda_j(c)x} \to 1$, $x \to -\infty$, for each $s \in [-h, 0]$. Then also it holds that $u(t, x - ct)e^{-\lambda_j(c)x} \to 1$, $x \to -\infty$, for each $t \geq 0$.*

Proof. Due to a step by step argument, it is sufficient to consider the situations when $t \in [0, h]$. Set $U(t,x) := e^t u(t,x)$, then $U(s, x - cs)e^{-\lambda_j(c)x} \to e^s$, $x \to -\infty$, and

$$U_t(t,x) = U_{xx}(t,x) + e^t g(e^{-t+h} U(t-h, x)), \quad t > 0, \; x \in \mathbb{R}.$$

Hence, by Duhamel's formula (see e.g. [15, Theorem 12, p. 25]),

$$U(t,x) = \Gamma(t,\cdot) * U(0,\cdot) + \int_0^t \Gamma(t-s,\cdot) * e^s g(e^{-s+h}U(s-h,\cdot))ds,$$

$$\text{where} \qquad \Gamma(t,x) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}, \ t>0, \ x \in \mathbb{R},$$

is the fundamental solution and $\Gamma(t,\cdot) * U(s,\cdot)$ denotes the convolution on $\mathbb{R}$ with respect to the missing space variable.

By Lebesgue's dominated convergence theorem, for each $s \in [-h,0], t>0$,

$$\lim_{x\to-\infty} e^{-\lambda_j x}\Gamma(t,\cdot) * U(s,\cdot) =$$

$$\frac{1}{2\sqrt{t\pi}} \int_{\mathbb{R}} e^{-\frac{1}{4t}[(y+2t\lambda_j)^2-4t^2\lambda_j^2]} \lim_{x\to-\infty} e^{-\lambda_j(x-y)}U(s,x-y)dy = e^{\lambda_j^2 t+\lambda_j cs+s}.$$

Consequently, for $t \in (0,h]$, we have that

$$\lim_{x\to-\infty} e^{-\lambda_j x}U(t,x) = e^{\lambda_j^2 t} + \int_0^t \lim_{x\to-\infty} e^{-\lambda_j x}\Gamma(t-s,\cdot) * g(e^{-s+h}w(s-h,\cdot))e^s ds$$

$$= \ e^{\lambda^2 t} + g'(0)e^{-\lambda ch}e^{\lambda^2 t}\int_0^t e^{(-\lambda^2+\lambda c+1)s}ds = e^{(1+\lambda c)t}.$$

Finally, we obtain the relation $\lim_{x\to-\infty} e^{-\lambda_j x}u(t,x) = e^{\lambda_j ct}$ for each $t \in (0,h]$ which completes the proof of the lemma. □

Remark 6. An obvious modification of the above proof yields the following assertion: *Assume that the birth function* $g : \mathbb{R}_+ \to \mathbb{R}_+, \ g(0) = 0,$ *is bounded and Lipschitz continuous. Suppose also that the initial fragments* $u_k(s,z), \ k = 1,2,$ *of bounded solutions* $u_k(t,z)$ *to equation (1.1) satisfy, for some positive* μ*, the relation*

$$(u_1 - u_2)(s, x-cs)e^{-\mu x} \to 0, \ x \to -\infty, \ s \in [-h,0].$$

Then $(u_1-u_2)(t,x-ct)e^{-\mu x} \to 0, \ x \to -\infty,$ *for each* $t \geq 0$.

This result provides a short and elementary justification for one of delicate moments in getting *a priori* estimates for a weighted energy method developed by Mei *et al.* [37, 38, 39, 40]. Indeed, an important initial fragment of the derivation of these estimates includes elimination of the boundary term

$$(u-\phi)(t,x-ct)e^{-\mu x}|_{x=-\infty} = (w(t,x)-\phi(x))e^{-\mu x}e^{\mu ct}|_{x=-\infty} = 0.$$

For instance, see [29, p. 855], [39, formulas (3.9)-(3.11)] or [28, p. 1067].

3.4 Proof of Theorem I.4

We will need the following result

Lemma 5. *Assume that the initial function $w_0(s,z) \geq 0$ is uniformly bounded on the strip $[-h,0]\times\mathbb{R}$ (say, by some $K>0$) and satisfies the hypothesis $(IC3)$ and, for some $c > c_*$, it holds*

$$\lim_{z\to-\infty} w_0(s,z)e^{-\lambda_1(c)z} = 1$$

uniformly on $s\in[-h,0]$. Then for each $\varsigma>0$ there exists L and $\psi(z) = (1+\varsigma+o(1))e^{\lambda_1(c)z}$, $z\to-\infty$, such that $\psi'(z)>0$ for $z\in\mathbb{R}$, $\psi(L)=K$, $\psi(+\infty)=+\infty$, $w_0(s,z)<\psi(z),\ z\leq L, s\in[-h,0]$, and

$$\psi''(z) - c\psi'(z) - \psi(z) + g(\psi(z-ch)) \leq 0,\ z\leq L. \tag{3.12}$$

Proof. Since $c>c_*$, the linearisation of equation (3.12) about 0 has exactly two real simple eigenvalues $\lambda_1(c)<\lambda_2(c)$. In particular, the linearised equation has a positive solution $(\phi(t),\phi'(t)) = (1,\lambda_2(c)))e^{\lambda_2(c)t}$. Moreover, the eigenvalue $\lambda_2=\lambda_2(c)$ is dominant (i.e. $\Re\lambda_j(c)<\Re\lambda_2$ for all other eigenvalues $\lambda_j(c), j\neq 2$). As a consequence, equation (3.12) has a solution $\psi_2(t)$ with the following asymptotic behaviour at $-\infty$:

$$(\psi_2(t),\psi_2'(t)) = (1,\lambda_2))e^{\lambda_2 t} + O(e^{(\lambda_2+\epsilon)t}),\ t\to-\infty,\ \epsilon>0,$$

(see e.g. [12, Theorem 2.1] for more detail).

In this way, there exists a maximal open non-empty interval $(0, T)$, $T \in \mathbb{R} \cup \{+\infty\}$, such that $\psi_2(t) > 0$, $\psi_2'(t) > 0$ for all $t \in (0, T)$. We claim that $\psi_2(T) > \kappa$ and $T = +\infty$. First, it should be noted that $\psi_2(T) \neq \kappa$ since otherwise we obtain a) if T is finite then $\psi_2(T) = \kappa > g(\psi_2(T - ch))$, $\psi_2'(T) = 0$, $\psi_2''(T) \leq 0$, contradicting to (3.12); b) if $T = +\infty$ then $\psi_2(t)$ is a monotone heteroclinic connection between 0 and κ, different from ψ_1. This contradicts to the uniqueness of the wavefront ψ_1 established in [55]. Next, suppose that $\psi_2(T) < 1$ and consider the difference $\theta_a(t) = \psi_1(t) - \psi_2(t + a)$, $t \in \mathbb{R}$, for some fixed $a \in \mathbb{R}$. Here $\psi_1(t)$ denotes the unique monotone wavefront to (3.12) normalised by the condition $\psi_1(t)e^{-\lambda_1 t} = 1 + o(1)$, $t \to +\infty$. Since ψ_1 is a strictly monotone heteroclinic connection between 0 and κ, there exists a unique $S \in \mathbb{R}$ such that $\psi_1(S) = \psi_2(T)$. Now, taking into account the inequality $\lambda_1 < \lambda_2$, we obtain that, for each fixed a, the function $\theta_a(t)$ is positive in some maximal interval $(-\infty, \sigma(a))$. If we choose $b = T - S$ then $\theta_b(S) = 0$, $\theta_b'(S) > 0$ and therefore $\sigma(b) = \sigma(T - S) < S$, $\theta_b(\sigma(b)) = 0$. On the other hand, $\theta_{a_1}(t) > 0$, $t \in [\sigma(b), S]$, for some large negative $a_1 \leq b$. Note also that $\theta_a(t) > \theta_b(t) > 0$, $t \leq \sigma(b)$ if $a < b$. In consequence, there exists $d \in (a_1, b]$ such that $\theta_d(\sigma(d)) = \theta_d'(\sigma(d)) = 0 \leq \theta_d''(\sigma(d))$. However, this yields the following contradiction:

$$0 = \theta_d''(\sigma(d)) - c\theta_d'(\sigma(d)) - \theta_d(\sigma(d)) + g(\psi_1(\sigma(d) - ch)) - g(\psi_2(d + \sigma(d) - ch)) > 0$$

because $\theta_d(\sigma(d) - ch) = \psi_1(\sigma(d) - ch) - \psi_2(d + \sigma(d) - ch) > 0$ and g is strictly increasing.

Finally, if $T < +\infty$ and $\psi_2(T) > \kappa$, then $g(\psi_1(T - ch)) < g(\psi_1(T)) < \psi_1(T)$. Since, in addition, $\psi_2''(T) \leq \psi'(0) = 0$, we obtain the following contradiction:

$$0 = \psi_1''(T) - c\psi_1'(T) - \psi_1(T) + g(\psi_1(T - ch)) < 0.$$

Next, we consider, for $\epsilon \in [0, 1]$ and $\mu \in (\lambda_1(c), \lambda_2(c))$, $\mu < (1 + \alpha)\lambda_1(c)$, the function

$$\psi(t,\epsilon) = \psi_2(t) + \epsilon(e^{\lambda_1 t} + e^{\mu t}).$$

It is clear that $\psi(t,\epsilon) \le Ce^{\lambda_1 t},\ t \ge 0$, for some $C > 1$ which does not depend on $\epsilon \in [0,1]$.

Set $\chi(z) = z^2 - cz - 1 + g'(0)e^{-zch}$, we have that

$$\mathcal{D}\psi := \psi''(t,\epsilon) - c\psi'(t,\epsilon) - \psi(t,\epsilon) + g(\psi(t-ch,\epsilon)) =$$

$$\epsilon\chi(\mu)e^{\mu t} + g(\psi(t-ch,\epsilon)) - g(\psi(t-ch,0)) - g'(0)\epsilon(e^{\lambda_1(t-ch)} + e^{\mu(t-ch)t}).$$

Let $T_0 < 0$ be such that $\psi(t-ch,\epsilon) \le \delta_0 := \psi(T_0,1)$ for all $t \le T_0,\ \ \epsilon \in [0,1]$. Then, for some $P(t,\epsilon) \in [\psi(t-ch,0), \psi(t-ch,\epsilon)]$, it holds that

$$|g(\psi(t-ch,\epsilon)) - g(\psi(t-ch,0)) - g'(0)\epsilon(e^{\lambda_1(t-ch)} + e^{\mu(t-ch)t})| =$$

$$|g'(P(t,\epsilon)) - g'(0)|\epsilon(e^{\lambda_1(t-ch)} + e^{\mu(t-ch)t}) \le$$

$$\epsilon(\psi(t-ch,\epsilon))^{\alpha}|(e^{\lambda_1(t-ch)} + e^{\mu(t-ch)t}) \le 2C\epsilon e^{(1+\alpha)\lambda_1 t}.$$

Thus, for a sufficiently large negative $T_1 < T_0$,

$$\mathcal{D}\psi \le \epsilon e^{\mu t}(\chi(\mu) + 2Ce^{[(1+\alpha)\lambda_1 - \mu]t}) < 0$$

for all $\epsilon \in (0,1]$, $t \le T_1$. As a consequence, if we define $\psi_\epsilon(t)$ by

$$\psi_\epsilon(t) := \begin{cases} \psi(t,\epsilon), & 0 \le t \le T_1, \\ y(t,\epsilon), & T_1 \le t, \end{cases}$$

where $y = y(t,\epsilon),\ t \ge T_1$, solves the initial value problem $y(s,\epsilon) = \psi(s,\epsilon),\ \ s \in [T_1 - h, T_1]$, $y'(T_1,\epsilon) = \psi'(T_1,\epsilon)$ for equation (3.12), then $\psi_\epsilon \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{T_1\})$ and $\mathcal{D}\psi_\epsilon(t) \le 0, t \ne T_1$. Define T_K as the unique solution of the equation $\psi_2(T_K) = K$, then due to the smooth dependence on the initial data,

$$(y(t,\epsilon), y'(t,\epsilon)) \to (\psi_2(t), \psi_2'(t)), \quad \epsilon \to 0+,$$

uniformly for $t \in [T_0, T_K]$.

Finally, due to the assumptions imposed on w_0, there exists $T_2 < T_1$ such that

$$w_0(t,s) \leq (1+\varsigma)e^{\lambda_1(c)z} < \psi_2(T_1),\ t \leq T_2,\ s \in [-h,0].$$

For $\epsilon \in (0,1]$, set $p_\epsilon = \lambda_1^{-1}(c)\ln[(1+\varsigma)/\epsilon]$ and $\tilde{\psi}(t) := \psi_\epsilon(t+p_\epsilon)$. Obviously, $\tilde{\psi}(t) > (1+\varsigma)e^{\lambda_1(c)z}$, $t \leq T_1 - p_\epsilon$, $\tilde{\psi}(t) > \psi_2(T_1)$, $t \in [T_1 - p_\epsilon, T_K - p_\epsilon]$ and $\tilde{\psi}(t) = (1+\varsigma+o(1))e^{\lambda_1(c)z}$, $t \to +\infty$. Since $\tilde{\psi}(T_K - p_\epsilon) = y(T_K, \epsilon) > K$, we obtain that

$$w_0(s,z) \leq \tilde{\psi}(z), \quad s \in [-h,0].$$

whenever $T_K < T_2 + p_\epsilon$. □

Next, for the solution $w(t,z)$ of the initial value problem $w(s,z) = w_0(s,z)$, $(s,z) \in [-h,0] \times \mathbb{R}$, we define its $\omega-$limit set by

$$\Omega(w_0) = \{w_* \in C^{1,2}([-h,0]\times\mathbb{R}) : \text{there exists some } t_k \to +\infty \text{ such that}$$

$$\lim_{k\to\infty} w(t_k+s,z) = w_*(s,z) \text{ uniformly on compact subsets of } [-h,0]\times\mathbb{R}\}.$$

Note that the set $\Omega(w_0)$ is non-empty, compact and invariant with respect to the flow generated by equation (3.1), e.g. see [51, Lemma 2.8].

Theorem 2. *Assume that the initial function $w_0(s,z) \geq 0$ satisfies the hypotheses $(IC1')$, $(IC2')$ and, for some $A > 0$ and $c > c_*$, it holds*

$$\lim_{z\to-\infty} w_0(s,z)e^{-\lambda_1(c)z} = A$$

uniformly on $s \in [-h,0]$. Choose a shifted copy of the wavefront profile ϕ normalised by the boundary condition $\lim_{z\to-\infty} e^{-\lambda_1(c)z}\phi(z) = 1$. Then

$$\lim_{t\to\infty} |\phi(\cdot+a) - w(t,\cdot)|_{\lambda_1} = 0,$$

where $a = (\lambda_1(c))^{-1}\ln A$.

Proof. Without loss of generality, we may assume that $A = 1$ (otherwise we can take a shifted copy of w_0). For each $\varsigma > 0$ there exist L and ψ satisfying all conclusions of Lemma 5 and such that $w_0(s,z) \le \psi_+(z)$, $(s,z) \in [-h,0]\times\mathbb{R}$, where

$$\psi_+(z) := \begin{cases} \psi(z), & 0 \le z \le L, \\ \kappa + |w_0|_\infty, & L \le z. \end{cases}$$

Since $-K + g(K) < 0$ and $\psi'_+(L-) > 0 = \psi'_+(L+)$, we conclude that $\psi_+(z)$ is a super-solution for equation (3.1). In view of Lemma 1, we also find that

$$w(t,z) \le \psi_+(z), \quad (t,z) \in \mathbb{R}_+ \times \mathbb{R}.$$

On the other hand, it is easy to see (e.g., cf. [57, p. 478]) that there exists a monotone C^1-function $\hat{g} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the hypothesis (**H**) and such that $g'(0) = \hat{g}'(0) \ge \hat{g}'(x)$, $g(x) \ge \hat{g}(x)$ for all $x \in [0,\kappa]$. Let $\hat{w}(t,z)$, $t > 0, z \in \mathbb{R}$, solve the initial value problem

$$w_t(t,z) = w_{zz}(t,z) - cw_z(t,z) - w(t,z) + \hat{g}(w(t-h,z-ch)), \tag{3.13}$$

$$w(s,z) = w_0(s,z),\ s \in [-h,0],\ z \in \mathbb{R},$$

then clearly $w(t,z)$ is a super-solution for (3.13) and therefore Lemma 1 implies that $\hat{w}(t,z) \le w(t,z)$ for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$. Furthermore, Theorem 1A assures that $\lim_{t\to+\infty} |\hat{w}(t,\cdot) - \hat{\phi}_c(\cdot)|_{\lambda_1} = 0$ for the wavefront $\hat{\phi}$ of equation (3.13) which is normalised as $\lim_{z\to-\infty} e^{-\lambda_1(c)z}\hat{\phi}(z) = 1$.

Next, let $w_u(t,z)$, $t > 0, z \in \mathbb{R}$, denote the solution of the initial value problem $w_u(s,z) = \psi_+(z)$, $s \in [-h,0]$, $z \in \mathbb{R}$, for equation (3.1). Then Corollary 9 implies that

$$\hat{w}(t,z) \le w(t,z) \le w_u(t,z),\ (t,z) \in \mathbb{R}_+ \times \mathbb{R}. \tag{3.14}$$

Therefore it holds, for some $a_1 \in [0, \lambda^{-1}(c)\ln(1+\varsigma)]$ and for all $w_l \in \Omega(w_0)$, that

$$\hat{\phi}(z) \le w_l(s,z) \le \phi(z+a_1),\ z \in \mathbb{R},\ s \in [-h,0], \tag{3.15}$$

where

$$1 = \lim_{z\to-\infty} \hat{\phi}(z)e^{-\lambda_1 z} \le \lim_{z\to-\infty} \phi(z+a_1)e^{-\lambda_1 z} \le \lim_{z\to-\infty} \psi_+(z)e^{-\lambda_1 z} = 1+\varsigma.$$

Next, since $\hat{\phi}(z)$ is a sub-solution for equation (3.1) we find analogously that, for some $a_0 \in [0, a_1]$ and for all $w_{ll} \in \Omega(w_l) \subset \Omega(w_0)$,

$$\phi(z+a_0) \le w_{ll}(s,z) \le \phi(z+a_1),\ z \in \mathbb{R},\ s \in [-h,0],$$

where

$$1 \le \lim_{z\to-\infty} \phi(z+a_0)e^{-\lambda_1 z} \le \lim_{z\to-\infty} \phi(z+a_1)e^{-\lambda_1 z} \le 1+\varsigma.$$

Since the latter relation holds for every $\varsigma > 0$, we conclude that actually $a_0 = 0$ and $\{\phi(\cdot)\} = \Omega(w_l) \subset \Omega(w_0)$. Furthermore, as a consequence of (3.15), $\lim_{z\to-\infty} e^{-\lambda z} w_l(s,z) = 1$ uniformly in $s \in [-h,0]$.

Hence, for each $\varsigma > 0$ there are $Z_1(\varsigma),\ T_\varsigma > 0$ such that, for all $t \ge T_\varsigma$, $z \le Z_1(\varsigma)$, it holds

$$\begin{aligned} &-2\varsigma \le e^{-\lambda_1 z}(\hat{w}(t,z) - \hat{\phi}(z)) - e^{-\lambda_1 z}(\phi(z) - \hat{\phi}(z)) \le \\ &e^{-\lambda_1 z}(w(t,z) - \phi(z)) \le e^{-\lambda_1 z}(\psi_+(z) - \phi(z)) < 2\varsigma. \end{aligned} \tag{3.16}$$

In addition, $\{\phi(\cdot)\} \in \Omega(w_0)$ implies that there exits a sequence $t_n \to +\infty$ that $w(t_n+s,z) \to \phi(z)$ on compact subsets of Π_0. This fact, together with (3.14) and (3.16), implies that

$$\sup_{s\in[-h,0]} |\phi(\cdot) - w(t_n+s,\cdot)|_{\lambda_1} \le 2\varsigma$$

for all sufficiently large n. Finally, an application of Corollary 13 completes the proof. □

Finally, we will apply Theorem I.4 in order to analyse behaviour of solutions whose initial data satisfy the hypotheses $(IC1')$, $(IC2')$ and (1.11):

Proof. of Corollary 1:

Case I: $\lambda > \lambda_*$. The statement of the corollary is an immediate consequence of [51, Theorem 1.4].

Case II: $\lambda < \lambda_*$. Clearly, we have that $\lambda = \lambda_1(c(\lambda))$. Set $A_- = \min_{s\in[-h,0]} A(s)e^{-\mu s}$. Then for each $A_1 < A_-$, the initial datum

$$w_1(s,x) := \min\{A_1 e^{\lambda(x+cs)}, w_0(s,x)\}$$

meets all the conditions of Theorem 1. Consequently, for each $\delta > 0$ there exists $T_\delta > 0$ such that solution $u_1(t,x)$ of the initial value problem $u_1(s,x) = w_1(s,x),\ (s,x) \in \Pi_0$, to equation (1.1) satisfies

$$\phi(x + ct + a_1) - \delta\eta_\lambda(x + ct) \le u_1(t,x), \quad \text{for all} \ \ x \in \mathbb{R},\ t > T_\delta$$

with $a_1 = \lambda^{-1}\log(A_1)$. Now, the functions ϕ and η_λ are equivalent at $-\infty$ so that, to each given $\epsilon > 0$ we can associate A_1 close to A_- and $\delta > 0$ close to 0 such that

$$(1-\epsilon)\phi(x + ct + a_-) \le u_-(t,x) \le u(t,x), \quad x \in \mathbb{R},\ t > T_\delta.$$

The upper estimation can be established in a similar way by comparing $u(t,x)$ with solution $u_2(t,x)$ of (1.1) satisfying the initial condition

$$w_2(s,x) = \max\{A_2 e^{\lambda(x+cs)}, w_0(s,x)\}, \quad (s,x) \in \Pi_0,$$

with $A_2 > A_+ = \max_{s\in[-h,0]} A(s)e^{-\mu s}$.

Case III: $\lambda = \lambda_*$. Inequalities (1.12) can be proved in the same manner as in Case II, if we take the initial functions

$$\tilde{w}_1(s,x) := \min\{A_1 e^{M(x+cs)}, w_0(s,x)\},\ \ \tilde{w}_2(s,x) = \max\{A_2 e^{\nu(x+cs)}, w_0(s,x)\},$$

where $\nu < \lambda_* < M$, instead of $w_1(s,x)$ and $w_2(s,x)$.

Now, inequalities (1.12) also imply that the only wavefront to which $u(t,x)$ can converge (as $t \to +\infty$) is some translation $\phi_*(x + c_*t + b)$ of the critical wavefront $\phi_*(x + c_*t)$. However, this is not possible in view of the following argument. Take some $A_1 < A_-$ and $\hat{g} \leq g$ satisfying (**H**) with $L_{\hat{g}} = g'(0)$. Set

$$w_*(s,x) = \min\{A_1 e^{\lambda_*(x+cs)}, w_0(s,x)\}.$$

Then by the comparison principle, solution $w_*(t,x)$ of the initial value problem

$$w_t(t,x) = w_{zz}(t,x) - w(t,x) + \hat{g}(w(t-h,x)),\ w(s,x) = w_*(s,x),\ (s,x) \in \Pi_0,$$

satisfies $w_*(t,x) \leq u(t,x)$ for all $t \geq 0$, $x \in \mathbb{R}$. On the other hand, by invoking Theorem I.5, we find that $w_*(t,x)$ converges uniformly to some wavefront $\hat{\phi}_*(x+c_*t)$ of the modified equation. Keeping $z = x + c_*t$ fixed and passing to the limit in $w_*(t,x) \leq u(t,x)$ (as $t \to +\infty$) for each fixed z, we find that $\hat{\phi}_*(z) \leq \phi_*(z+b)$ for all $z \in \mathbb{R}$. However, this is not possible since $\phi_*(z)$ decays at $-\infty$ faster than $\hat{\phi}_*(z)$.

Finally, in order to prove the inequality (1.13), it suffices to consider the initial function

$$\tilde{w}_3(s,x) = \max\{-xe^{\lambda_*(x+cs)}, w_0(s,x)\},\ (s,x) \in \Pi_0,$$

instead of $w_2(s,x)$. Then we proceed can similarly to the proof of inequalities (1.12) by applying Theorem I.5A. $\square$

3.5 Proof of Theorem I.6 and Corollary 3

Let the triple $(c, \lambda_c, \gamma) \in [c_\#, +\infty) \times [\lambda_1(c), \lambda_2(c)) \times \mathbb{R}_+$ be as in Lemma 2 (i.e. $\lambda_c = \lambda_1, \gamma = 0$ if $c = c_\#$ and $\gamma > 0, \lambda_c \in (\lambda_1(c), \lambda_2(c))$ if $c > c_\#$). Theorem I.6 and Corollary 3 follow from the next assertions.

Lemma 6. *Assume (**UM**) and let the initial function w_0 satisfy (IC1). Consider $c \geq c_{\#}$ and let $\phi(z)$ denote a positive semi-wavewfront to equation (3.2). Then the inequalities*

$$\phi(z) - qe^{-\gamma s}\xi(z-b,\lambda_c) \leq w_0(s,z) \leq \phi(z) + qe^{-\gamma s}\xi(z-b,\lambda_c), \ (s,z) \in \Pi_0,$$

(where $q > 0, b \in \mathbb{R}$ are some fixed numbers) imply that the solution $w(t,z)$ of (3.1), (3.2) satisfies

$$(3.17) \quad \phi(z) - qe^{-\gamma t}\xi(z-b,\lambda_c) \leq w(t,z) \leq \phi(z) + qe^{-\gamma t}\xi(z-b,\lambda_c), \ t \geq 0, \ z \in \mathbb{R}.$$

Proof. Set $\delta_{\pm}(t,z) = \pm(w(t,z) - (\phi(z) \pm qe^{-\gamma t}\xi(z-b,\lambda_c)))$ and

$$(\mathcal{L}\delta)(t,z) := \delta_{zz}(t,z) - \delta_t(t,z) - c\delta_z(t,z) - \delta(t,z).$$

Then

$$(\mathcal{L}\delta_{\pm})(t,z) = \mp(g(w(t-h,z-ch)) - g(\phi(z-ch))) + qe^{\lambda_c(z-b)}e^{-\gamma t}[-\lambda^2 + c\lambda + 1 - \gamma].$$

Therefore we obtain, for all $z \in \mathbb{R}$, $t \in (0,h]$,

$$(\mathcal{L}\delta_{\pm})(t,z) \geq qe^{\lambda_c(z-b)}e^{-\gamma t}[-\lambda^2 + c\lambda + 1 - \gamma - g'(0)e^{\gamma h}e^{-ch\lambda_c}] \geq 0.$$

Since, in addition, $\delta_{\pm}(0,z) \leq 0$ and $\delta(t,z)$ is exponentially bounded, an application of the Phragmèn-Lindelöf principle yields $\delta_{\pm}(t,z) \leq 0$ for all $t \in [0,h]$. Finally, the step by step procedure completes the proof of the inequality $\delta_{\pm}(t,z) \leq 0$ for all $t \geq 0$. □

Lemma 7. *Let all the conditions of Lemma 6 be satisfied and $c \geq c_{\#}$. Assume, in addition, that $|g'(u)| < 1$ on some interval $[\kappa - \rho, \kappa + \rho]$, $\rho > 0$. If, for some $b \in \mathbb{R}$, the semi-wavefront profile satisfies: $\phi(z-ch) \in (\kappa - \rho/2, \kappa + \rho/2)$, for $z \geq b$. Then ϕ is actually a wavefront (i.e. $\phi(+\infty) = \kappa$) and*

$$|w_0(s,z)-\phi(z)| \le q\eta_{\lambda_c}(z-b),\ (s,z)\in \Pi_0, \tag{3.18}$$

for $q \in (0, \rho/2]$ *implies that the solution* $w(t,z)$ *of (3.1), (3.2) satisfies*

$$|w(t,z)-\phi(z)| \le qe^{-\gamma t}\eta_{\lambda_c}(z-b),\ t\ge 0,\ z\in\mathbb{R}. \tag{3.19}$$

Proof. We only make the upper estimation, the lower estimation is analogous. We denote $\delta(t,z) = w(t,z) - (\phi(z) + qe^{-\gamma t}\eta(z-b))$.

So, for $(t,z)\in[0,h]\times(-\infty,b]$ we have:

$$\begin{aligned}
(\mathcal{L}\delta)(t,z) &= (\mathcal{L}w)(t,z) - (\mathcal{L}\phi)(z) - q\mathcal{L}(e^{-\gamma t}\eta(z-b)) \\
&= g(\phi(z-ch)) - g(w(t-h,z-ch)) + qe^{\lambda(z-b)}e^{-\gamma t}[-\lambda^2 + c\lambda + 1 - \gamma] \\
&\ge -L_g|\phi(z-ch) - w(t-h,z-ch)| + qe^{\lambda(z-b)}e^{-\gamma t}[-\lambda^2 + c\lambda + 1 - \gamma] \\
&\ge -g'(0)qe^{-\gamma(t-h)}\eta(z-ch-b) + qe^{\lambda(z-b)}e^{-\gamma t}[-\lambda^2 + c\lambda + 1 - \gamma] \\
&\ge qe^{\lambda(z-b)}e^{-\gamma t}[-\lambda^2 + c\lambda + 1 - \gamma - g'(0)e^{\gamma h}e^{-\lambda ch}] \ge 0
\end{aligned}$$

And for $(t,z)\in[0,h]\times[b,\infty)$ we have:

$$\kappa - \delta_0 \le \phi(z-ch) - qe^{-\gamma(t-h)}\eta(z-b-ch) \le w(t-h,z-ch) \tag{3.20}$$

$$w(t-h,z-ch) \le \phi(z-ch) + qe^{-\gamma(t-h)}\eta(z-ch-b) \le \kappa + \delta_0 \tag{3.21}$$

Now, for some $\theta(t,z)\in[\kappa-\delta_0, \kappa+\delta_0]$ we have:

$$\begin{aligned}
(\mathcal{L}\delta)(t,z) &= (\mathcal{L}w)(t,z) - (\mathcal{L}\phi)(z) - q\mathcal{L}(e^{-\gamma t}\eta(z-b)) \\
&= g(\phi(z-ch)) - g(w(t-h,z-ch)) + qe^{-\gamma t}[1-\gamma] \\
&\ge g'(\theta(t,z))[\phi(z-ch) - w(t-h,z-ch)] + qe^{-\gamma t}[1-\gamma] \\
&\ge g'(\theta(t,z))qe^{-\gamma(t-h)} + qe^{-\gamma t}[1-\gamma] \ge 0
\end{aligned}$$

So maximum principle implies:

$$w(t,z) \leq \phi(z) + qe^{-\gamma t}\eta(z-b) \quad z \in \mathbb{R}\ t \in [0,h]$$

So, using (3.20), (3.21) and step method on intervals $[h,2h],[2h,3h]$... we obtain (3.19).

Finally, since $g : [\kappa-\rho,\kappa+\rho] \to [\kappa-\rho,\kappa+\rho] =: \mathcal{I}$ is well defined and

$$\kappa - \rho \leq m := \liminf_{z to +\infty} \phi(z) \leq M := \limsup_{z to +\infty} \phi(z) \leq \kappa + \rho,$$

it follows from [19, Remark 12] that $g([m,M]) \supseteq [m,M]$. On the other hand, g is a contraction on $\mathcal{I}$ so that $M = m = \kappa$. □

Lemma 8. *Let $g(x)$ and $w_0(t,z)$ meet all the assumptions of Corollary 3. Then inequality (3.8) implies that the solution $w(t,z)$ of (3.1), (3.2) satisfies (3.19) for some positive ρ, γ.*

Proof. It is easy to see that there exist monotone functions $g_+, g_- : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(i) $g_-(x) \leq g(x) \leq g_+(x),\ x \geq 0$;

(ii) $g_-(x) = g(x) = g_+(x)$ for all x from some neighborhood of 0;

(iii) g_+ satisfies **(H)** with $\kappa_+ = g(x_m)$ and g_- satisfies **(H)** with $\kappa_- = g(g(x_m))$.

Let $w_\pm(t,z)$ denote the solution of the initial value problem

$$w_t(t,z) = w_{zz}(t,z) - cw_z(t,z) - w(t,z) + g_\pm(w(t-h, z-ch)),$$

$$w_\pm(s,z) = w_0(s,z),\ (s,z) \in \Pi_0,$$

and let $\phi_\pm$ be wavefront solutions of the stationary equations

$$0 = \phi''(z) - c\phi' z - \phi(z) + g_\pm(\phi(z-ch)).$$

normalized by the condition $\lim_{x\to-\infty} \phi_\pm(x)/\phi(x) = 1$ (this is possible in view of (iii)). Then Theorem I.5A (applied to $w_\pm(t,z)$) and the comparison principle guarantees that, for each small $\epsilon > 0$ there exist a large $b > 0$ and $T > 0$ such that

$$(3.22)\quad \kappa_- - \epsilon \le w_-(t,z) \le w(t,z) \le w_+(t,z) \le \kappa_+ + \epsilon, \text{ for all } t \ge T,\ z \ge b - ch.$$

Henceforth, we will fix $\epsilon > 0$ small enough to have $|g'(x)| < 1$ for all $x \in [\kappa_- - \epsilon, \kappa_+ + \epsilon]$.

Now, invoking Lemma 6, we also obtain that

$$(3.23)\qquad |w(t,z) - \phi(z)| \le 0.5\rho e^{-\gamma t} e^{\lambda(z-b)},\ t \ge T, z \le b,$$

with $\rho < 0.5(k_+ - k_-)$.

Finally, using relations (3.22), (3.23) and Lemma 7 we obtain the proof of the lemma. □

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